THE TEACHING
OF
ELEMENTARY MATHEMATICS

THE TEACHING OF ELEMENTARY MATHEMATICS

by

CHARLES GODFREY

and

A. W. SIDDONS

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PREFACE TO SECOND EDITION

Though this book was written some years ago, if it were being entirely rewritten to-day, the only change that would be necessary would be to note that some of the reforms suggested in the chapter "Outlook in Mathematics" have already been widely adopted. The rest of the book, particularly the detailed advice about teaching practice, is as up to date as ever it was.

At the present time there is an urgent demand for teachers: men and women from all walks of life and of widely differing ages are entering the recently opened Emergency Training Colleges. The Education Act of 1944 will, when fully implemented, change the whole aspect of the educational system of this country. The traditional Secondary Grammar Schools will be partnered by Secondary Modern and Secondary Technical Schools. Education is a word that is on everyone's lips and everywhere the future of the "new type secondary school" is being discussed. The question of what shall be taught in these schools cannot be dictated in advance—curricula must grow naturally. In the transition period which may well be one of ten years' duration the children must be taught. The subject matter of the course and the methods of teaching will both pass through an experimental stage but, in the main, both matter and method will be determined in the light of the experience of tried and proved teachers. New methods must be tried if stagnation is to be avoided, but every student in training will benefit by a careful study of the best methods in use at the present time.

This book was originally written for Preparatory and Secondary Grammar Schools. The outlook, however, is broad and the section dealing with "The Place of Mathematics in Education" is as valid to-day as it was when it was written, and applies to teaching at all stages and in all types of school. The part dealing with "General Teaching Points" is a direct statement of practical politics. Here the student will find no thrills and fancies but concise answers to the many questions that confront the teacher in the everyday work of the mathe-

matical classroom. Of the sections dealing with the various branches of mathematics those on arithmetic and algebra will be found useful in all types of school, but parts of the section on geometry will appeal mainly to those who will work in the secondary grammar schools.

A. W. SIDDONS

November 1945

PREFACE TO FIRST EDITION

This book deals with the Teaching of Mathematics from about the age of 9 up to the School Certificate stage. It thus covers the mathematical work of Preparatory Schools and the work of Secondary Schools up to the age of specialisation, except that it gives no detailed recommendations about Mechanics and Calculus.

The late Professor Godfrey and I planned the book over a dozen years ago; but other duties prevented us from carrying out our plan at once, as we had intended.

Part I was found among Professor Godfrey's papers after his untimely death in 1924.

The parts on Arithmetic and Algebra were written by me mainly between 1922 and 1924 and were criticised then by Professor Godfrey; these have been brought up to date, and the rest of the book has been written by me in the course of the past year.

The changes made in the teaching of Geometry nearly 30 years ago were, and to some extent still are, so little understood in some quarters that much of the teaching is aimless and formless; so I have treated the first two years of Geometry teaching in considerable detail. I have tried to give the work an aim: the course I have sketched is perfectly straightforward and has proved itself successful in the hands of many teachers.

To many friends, colleagues and pupils I owe much for conscious or unconscious suggestions. Throughout Parts II–VI I have had the advantage of Professor Godfrey's inspiration and of the use of notes, etc., which I found among his papers after his death; but I take entire responsibility for anything that may be criticised in those parts.

By the courtesy of the editor of the *Preparatory Schools Review* I have been able to make use of some articles which I wrote for that paper some years ago.

A. W. SIDDONS

August 1931

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PART I

THE PLACE OF MATHEMATICS IN EDUCATION

BY THE LATE

PROFESSOR CHARLES GODFREY

These chapters were found among the papers left by the late Professor Godfrey. They appear to have been written about 1911.

CHAPTER I

WHY TEACH MATHEMATICS?

The psychology of education is still in the making; till we know more of this science, it is wise to base our case for mathematics (or any other subject) rather on the commonsense ground of everyday experience than on far-reaching theoretical considerations.

In things of the mind, as in things of the body, we are still fairly ignorant of the effect of any particular course of diet: till we know a good deal more, our safest guide will be appetite. Now the idea that mathematics is the most repulsive of all subjects is so firmly imbedded in popular belief that it may seem hopeless to base any argument on appetite. But the experience of good teachers does not by any means confirm popular belief. The experience of good teachers is that a small minority of boys are almost incapable of mathematical reasoning, and derive no perceptible benefit from it: that another small minority take to the subject like ducks to water, and are happy in no other element: and that the great majority of boys can be led to find mathematics an interesting study, and to face a fair amount of solid work without grumbling. This is modern experience of mathematical teaching. Add to it a respectable record of antiquity, exceeding by a few centuries that of Latin Grammar, and the inference is reasonable that mathematics is an instrument of education found to be in conformity with the needs of the human mind.

II

Boys are sent to school in order that they may make intellectual efforts: if this is not true, it ought to be. There is no difficulty in setting tasks, in any subject, that call for effort. But effort is not all that is needed. Effort must lead to success:

otherwise, discouragement and apathy. Again, it is good that the boy should be able to judge of his own success. Now in these two respects mathematics has great advantages. The tasks can be graduated nicely to the powers of the worker: there is never any necessity to set hopeless tasks. At times, of course, the teacher may deliberately set too hard a problem: perhaps to lead up to a new method which makes the problem possible: or perhaps to correct intellectual pride. But this is an advanced touch, and ordinarily if the work is too hard for the boy, the fault is with the teacher, not with the subject.

Then, as to the boy judging his own success, there is a great difference between one subject and another. There is no wish here to depreciate literary studies; they have their special virtues which can never be replaced by anything that mathematics or science has to offer. And indeed, in such work as Latin prose composition, it can be made plain to the boy whether or no he has obeyed the rules of syntax. But, speaking generally, in literary work the verdict is a matter of taste and authority, the matured taste of the teacher or the authority of the classical author. The boy, as he develops, will learn, no doubt, to appreciate the criticisms offered, and the teacher will be successful in proportion as he gains the boy's assent; but the boy can hardly be his own judge. In mathematics to a great extent he can judge his own work. In problems of calculation he can be taught to apply checks to his work, which will tell him plainly if he has gone wrong. In geometry, the teacher can say "This step is wrong; why?" and the boy can satisfy himself that the step is wrong, and why it is wrong. In this way mathematical work is well fitted to develop independent and self-reliant habits of mind.

III

which characterises mathematics; this has always been accepted as a peculiar merit of the subject. There is no room for vagueness

of thought, the hiding place of the lazy mind: the boy must stand up to his difficulties: there is no escape for him.

How far the habit of precise thought in mathematics begets a habit of precise thought in other matters, is a question for the psychologists. Again, the mathematical specialist lives in a world of thought where everything is clear cut, and every question has its "Yes" or "No"; is the cast of mind so formed well fitted to deal with practical problems of life, where things refuse to be treated too absolutely? Is not the student of mathematics in need of a corrective to his too-great precision; such a corrective as history work, which calls into play judgment of men and balancing of probabilities?

Teachers of elementary work need not be moved by these speculations: they find themselves combating vagueness all day long, whatever subject they are teaching, and they accept mathematics as an ally.

It is worth remarking that the precision of mathematics belongs to its central and established portions. The foundations are, and always will be, a matter of controversy among mathematical philosophers. The axioms of Euclid are no longer accepted as the bed rock of geometry. The further back we push the starting-point, the more does it enter the region of metaphysics. For teaching purposes we have to choose arbitrarily some starting-point where the ground seems to be reasonably solid. From this point onward the course is clear till the student approaches the frontier of knowledge, where again he will lose the sharp dividing-line between true and false.

IV

Precision of thought is nearly akin to precision of language, and for teaching precision of language mathematics ranks second to law; that mathematics is a good preparation for law is shown by a glance at the Cambridge Tripos Lists. Every teacher of mathematics can be, and should be, a teacher of

English as well. Direct teaching of English should enter into the instruction of every one, and it is to be hoped that this obvious truth will soon be recognised even in the best schools. But direct teaching of English should never supplant indirect teaching. There must be special hours devoted to English: but English must still be taught incidentally at all other lessons given in the mother tongue. Different subjects will bring out different features; and the special feature that mathematics has to bring out is precision and unambiguity of language.

Mathematics gives scope for induction as well as deduction. The word "induction" is used here not in the special sense of "mathematical induction," but in the wider sense generally associated with the physical sciences. Geometry, in fact, is one of the physical sciences, and lends itself largely to inductive processes.

This point will be dealt with more fully when we have to discuss the methods of mathematical teaching: here it is enough to put in a claim that opportunities for induction and intuition and imagination are among the essential merits of mathematical study.

VI

Modern civilisation stands on a foundation of applied mathematics: without mathematics the earth could not support its present population. This statement will not be controverted by anyone familiar with the work of modern scientists and engineers, and with the part played by mathematics in their achievements.

The average man takes no direct part in these developments, but it is not fitting that he should live as a mere parasite on the organisation that keeps him alive. It is the work of the few to develop steam, electricity and machinery, to render navigation secure by predicting years ahead the motions of the

heavenly bodies, to join the continents together with steamship lines, electric cable and wireless, to cover new corn-bearing lands with railways, to tunnel the Alps and the Andes, to prepare for the day when our coal will fail by extracting electric power from water-falls and tides, finally, perhaps to revolutionise the world by controlling sub-atomic energy. These are some of the contributions of science to our means of existence; and mathematics is the tool that science uses. The modern man should have at least some conception of the means by which these results, so vital to him, are obtained. Geography will teach him what is being done and how, directly or indirectly, it is affecting his life. Mathematics and science will teach him how it is being done.

Here we are urging the "outlook" value of mathematics, rather than the utilitarian. We are assuming that the majority of boys will not make any direct use of mathematics in after life, that they will not even be able to follow in detail the mathematical methods of engineering and applied science. But they can be so taught that a vista is opened through which may be seen the tremendous potentialities of the study whose elements they are mastering. They should be brought to the stage from which a broad undetailed view may be obtained over the country of applied mathematics: and they may be shown the beginnings of a few of the roads that lead through this country. A public must be created able to realise what science and mathematics are doing for the world, and to form some general conception of the means.

VII

The average man will not be more than a spectator of the world's material progress: we have suggested that he may as well be taught to be an intelligent spectator. But the world needs an increasing number of workers trained to use mathematics; there must be specialists. Here for the first time we

come to the utilitarian argument. The world needs a certain number of mathematicians to do its work: and as the world is prepared to pay for this, a certain number of boys at school must be learning mathematics with a view to their future livelihood. The utilitarian argument is a perfectly respectable argument: we have refrained from putting it in the forefront, not through doubt as to its propriety, but because it applies to a relatively small proportion of boys and to a much smaller proportion of girls.

CHAPTER II

THE AIMS IN TEACHING MATHEMATICS

The preceding chapter was perhaps rather academic. It may be supposed to be addressed to an imaginary Board of Education, engaged in the task of setting up a system of education in a new country; a state of things unlikely to occur in England.

But perhaps it is good to question ourselves sometimes as to the reasons for our faith in mathematical teaching; and at any rate the preceding discussion clears the ground for the next enquiry: what are, and what should be, the aims of such teaching.

The aims of teachers of any subject seem capable of analysis into three more or less distinct elements: these we will call the "formal" aim, the Herbartian aim, and the utilitarian aim.

I

The "formal" aim shall be treated first, as it is the aim that has inspired (or failed to inspire as the case may be) probably the majority of schoolmasters. This theory of teaching has for its watch-words—"mental gymnastics," "moral and intellectual discipline." Its tendency is to emphasise the *process* of learning rather than the thing learnt; it does not matter much what you teach, what does matter is *how* you teach it; a postulate that will raise a responsive echo from every practical teacher.

A palpable embarrassment arises from the attitude of the boy, who is apt to care more about the "what" than the "how."

The working of the formal aim may be traced in the various subjects of instruction. In classics it lays stress on grammatical structure rather than on subject-matter, and ascribes great value to composition in the ancient languages. Consistently

with this view of language teaching, it imports the same method into instruction in modern languages and the mother tongue. History and geography, essentially outlook subjects, are not much favoured by the formalist school of teachers.

Mathematics has always been found very amenable to formalist treatment. It would be unfair to judge an educational aim in the light of its excesses, and indeed judged in this way the utilitarian aim would be in danger of equal condemnation. But it will be recognised that an excess of formalist zeal was responsible for the absurdities of the old geometry teaching—the demand for verbal reproductions of propositions (the figure to have the same lettering as in the book), the prohibition of algebraical symbols and of the use of dividers to transfer lengths, the tedious proofs of the obvious. The same spirit used to decree that, in certain examinations at Cambridge, candidates must eschew the methods of the calculus and cast their reasoning in geometrical language.

Now there can be no question of breaking altogether with the formal aim in education; however much we may value subject-matter and content, most of us do believe very profoundly that the method and form of instruction is a vital matter. But when we are invited to let the formal aim determine the subject-matter, there we have to part company. A certain topic—to avoid controversy, let us say poker-patience—is recommended for inclusion in the curriculum; it is admitted that the study of poker-patience does not widen the outlook, and that it does not necessarily form part of any correlated system of ideas; it is recommended as a mental gymnastic: and we find that by this is meant that poker-patience develops some faculty of the mind—let us say the faculty of arrangement. Now here lurks a psychological assumption. For it is obvious that no one would advocate the cultivation of this faculty through poker-patience if the faculty when cultivated will bear upon nothing but poker-patience. It

is assumed that there is a "spreading" effect; that anyone who has learnt to arrange his cards successfully at poker-patience will ever afterwards be the more competent to arrange things other than cards; to arrange, say, a railway time-table, or a battle-field, or a pageant. It is assumed that the ability will not remain specialised, but will become general. This assumption is a matter about which psychologists are becoming more and more doubtful. In the case of one faculty, that of memory, their opinion is now absolute. To practise remembering mathematical formulae does not improve the memory for poetry, or statistics, or maps. Perhaps it does improve the memory for other allied mathematical formulae: probably this depends on association. But there is said to be no spreading effect.

Take the "faculty" of observation. Some science teachers would have us believe that a person who has been trained to observe phenomena in laboratory will be observant of other things, will be an observant scout or an observant botanist. But psychologists will have none of it. We acquire the habit of observing chemical reactions through being continually on the *qui-vive* for that class of phenomena; smells perhaps, or precipitates, or changes of temperature; our interests are alert in that particular direction; the train is laid and a spark will touch it off; we know what to look out for; we instinctively disregard the irrelevant and seize the relevant.

But if we imagine that we can turn to the study of botany and find ourselves as observant in the fields as in the laboratory, we may be disappointed. We may pass over an orchid on the chalk downs and take it for a spike of flowering grass; we are not on the look-out for the discriminating features. Here is an actual instance of the specialised quality of the observation habit. I happen to be interested in botany, but am not particularly observant of other live things. I was once walking by the side of a hedge with a friend, and pointed out to him a dusky purple columbine flower in the middle of the hedge. He had not

noticed the flower, but drew my attention to a bird's nest near it, which I had failed to notice. Each of us had developed a specialised habit of observation, a habit that failed to spread.

Educational thought and literature is honey-combed with the assumption that it is possible to cultivate general faculties of the mind by specialised mental exercises. I am not prepared to say that the assumption is untrue; but, in the light of modern psychology, it seems a very unsafe assumption to build upon. Euclid is supposed to make us logical all round; but is it not probable that the habits of logical thought developed by the study of Euclid are excited solely when the object of thought is something akin to Euclid? For instance, to write Latin prose calls for a type of logical thought which, if analysed, would turn out to be unlike the type of thought required in geometrical reasoning. Are mathematical boys found to be strong on the logical side of Latin prose composition? Women are conventionally supposed to be illogical, which means I imagine that they are supposed to jump to conclusions, generally right conclusions. Are mathematical women less prone to this non-Euclidean process?

The study of classical literature is believed to cultivate the power of appreciating all literature. But we sometimes find that the more a man cares for Greek literature, the less he cares for English; presumably the beauties of the two languages are diverse; but we have another warning against loose assumptions of transferred educational effect.

The formal aim in education is based upon the assumption that the effects of training in one subject are transferred to others: hence the view that it does not matter greatly what subjects are taught. Certain mental faculties are to be developed that will be of universal application.

II

Sharply contrasted with the formalist view of education we have Herbart's view, charmingly set forth in Chapter III of Professor Adams' little book *Herbartian Psychology*. The central idea of this theory is one on which every competent teacher acts instinctively, however much he may glory in his ignorance of psychology. In a first lesson on vulgar fractions, you do not begin by writing down $\frac{3}{5}$; you begin by cutting up a cake—real or imaginary. The notation $\frac{3}{5}$ would leave the pupil cold; there is nothing in his mind to which $\frac{3}{5}$ would appeal; the idea would be a perfect stranger, and would gain a lodgment with difficulty, if at all. But the idea of cake, and of bits of a cake, are old-established frequenters of the mind; they can take the new idea by the hand and introduce it. Every good teacher in presenting a new idea seeks for some bit of pre-existing knowledge on to which he may tack the new idea. The mental process by which the old-established ideas take the new-comers by the hand is a perfectly familiar process. How many people took an interest in the Malay Peninsula before they associated it with the idea of rubber? The process is familiar, and it is a pity that a new word had to be invented to describe it. However, this could not be helped, and we may as well accept the word "apperception" without associating it with any prejudices we may happen to feel about training of teachers.

New matter has to be presented to the pupil in such a way that it may be apperceived, and woven into a mass with knowledge already gained. The Herbartian ideal then is not mere knowledge, but a mass of knowledge that hangs together and grows together; a mass of correlated ideas. Each bit of knowledge should illuminate some other bit; water-tight compartments, e.g. between algebra and geometry, must be done away with; different subjects, e.g. mathematics and physics and geography must be correlated. There will be all sorts of cross-lights and suggestions.

All of this has an obvious bearing on two other things—memory and interest.

Our chance of remembering any particular fact depends largely on the number of other facts with which we associate it; the only way of enlarging the memory is to multiply associations. This suggests an argument in favour of adding the experimental method to other methods of instruction; the pupil is piling up associations. And again, the more he correlates one subject with another, the more connecting threads of memory does he form; induced currents of thought, in fact. Patent memory systems depend on some set of arbitrary associations; numbers suggesting words, and so forth. The teacher who consciously appeals to apperception relies upon natural associations; e.g., he will approach the principle of Archimedes in the same way as did Archimedes, *via* the bath.

The bond between interest and apperception is equally obvious. Cricket helps us to apperceive the notion of an average. We are not as a rule interested in isolated scraps of information, unrelated to anything we know already. I am alive to the objection that the success of periodicals such as *Tit-Bits* points to a different conclusion. But the more educated a person is, the more he is bored by *Tit-Bits*. Children, it must be admitted, do take a wonderful interest in perfectly irrelevant facts; and perhaps there is more in heaven and earth than is dreamt of in our philosophy. Anyway, we teachers are committed to the view that *Tit-Bits'* way is not our way; we do find that children are interested in new matter that bears upon their existing knowledge and experience; that, as a rule, they are not interested in matter that they cannot apperceive, that they cannot bring into relation with their own world of thought. Very often, indeed, their minds seem to work in strange zig-zag paths, paths that education or the lapse of time has eliminated from the well-regulated, middle-aged mind of their teacher. Hence their mental associations may seem to us fantastic, and

we may conclude that purely isolated bits of knowledge appeal to their interests more than do our carefully planned overtures. But if we could get inside their mind, no doubt we should find that what seems to us irrelevant is to them just the reverse.

What is the bearing of all this upon curriculum-building? The formalist frames his curriculum with the object of developing certain faculties. The Herbartian allows the mental content to grow by laying hold of ideas that will hook on to the old. He will not be much moved by the contention that this or that bit of work will strengthen this or that intellectual muscle, for he does not view the mind as a muscular system. He will rather enquire what affinities there are between the new and the old. And the old will include not only old knowledge but old experience; he does not, in fact, discriminate between knowledge and the results of experience, for he holds that knowledge and experience really form a single whole.

III

If we listen to a mechanic explaining one of his machines to a class, we get a view of the utilitarian aim in its crudest form. He knows what knowledge he wants to impart, and he is convinced that it is useful. Of course he cares nothing about faculty-training; where he parts company from the Herbartians is that he pays no attention to the mental state of his hearers. He uses terms that they do not understand; he explains the details of his machine before he has put the class in possession of the general idea of its use and working.

I need not waste words in attacking crude utilitarianism, but there are one or two fairly obvious remarks that may be worth making. In the first place, a boy is a convinced utilitarian. If we wish to take him at a disadvantage, this is the side we must attack. We have to pretend to be utilitarians whatever mental reservations we may be making. If there is any chance of

getting him to believe that a new mathematical process is of practical use, we shall be wasting a strategic advantage if we are coy about it. The utilitarian appeal is sound psychology.

A second point is that we cannot often tell a boy honestly that mathematics is going to be directly useful to *him*. To most boys, at any rate in public schools, mathematics is not going to be a bread and butter subject. But there are not many who are quite so hard-headed as to make this a condition of taking an interest. If they can see that a piece of mathematics is of use to someone, this is a good enough introduction; and when once the interest is aroused, it will sustain itself for quite a long flight before coming to earth; then we must find another point of contact with real affairs and start off again. These contacts with reality make teaching easier; and these are important from the point of view indicated in Chapter I, namely in training up an educated public opinion which can appreciate the work of mathematicians, and in forming a social atmosphere in which mathematical study can live.

The above analysis of teaching aims is of course a mere sketch, and imperfect in many ways. There is another ideal that has not been classed as one of the chief competitors, because it actuates comparatively few teachers—the ideal of the systematic and scientifically complete presentation of a subject. There are three possible orders in which a subject can be presented; the historical order, the psychological order; and the scientific order. The historical order, the order of discovery, is the most definite of the three, and is usually scrappy and unsymmetrical. The psychological order, the order indicated by psychology as the most suitable, this we cannot know in our present state of ignorance; but it is probably more akin to the historical than to the scientific. By the scientific order is understood that order which is adopted by experts for the final statement of a worked-out subject. There is not one scientific order, but many: they tend to assume the deductive form, and

are as numerous as the possible sets of fundamental assumptions: and again, as no subject is really worked out, there is always room for reconsideration of values and proportions in the systematic statement adopted. But there is one feature common to all scientific orders; they are always an afterthought; the subject did not grow in that shape. They are like a completed engineering work divested of the scaffolding; marvellous, but less instructive now than in the rougher stage. Appreciation of a systematic presentation is a mark of maturity, and it is probably unnecessary to labour the point that the systematic order is not the order for teaching.

CHAPTER III

METHODS
DEDUCTION, INTUITION, INDUCTION

It is unnecessary to labour the point that mathematics gives excellent opportunity for deductive reasoning. Compare mathematics with geography, which nowadays almost claims to be a deductive science. Given latitude, winds and land relief, we deduce rainfall; given rainfall and soil we deduce distribution of vegetation and of life generally. But how often the deduction is vitiated by neglect of some complication in the data. Geography *can* be set out deductively, but a deductive study of geography is much more difficult than a deductive study of mathematics. If there is such a thing as training in deductive thought (much virtue in an *if*), mathematics seems to be the best medium for such training.

But mathematical truth is not essentially deductive. Given a body of truth, it may be stated in a deductive form; but the form is one thing and the body of truth is another. Deduction is a process of human thought. But I take it that the fact embodied in Pythagoras' theorem was true before human life appeared on this planet: if not, at what date was it first true? or did it become true gradually, as human thought developed?

Mathematical truth is not deductive, nor are mathematical truths discovered deductively. How mathematical discoveries are made we are told by Professor Hobson*.

"I have emphasised above the necessity and importance of fitting the results of mathematical research in their final form into a framework of deduction for the purpose of ensuring the complete precision and the verification of the various mathe-

* Presidential address to Section A of British Association, 1910, reported in *Nature*, September 1, 1910.

matical theories. At the same time, it must be recognised that the purely deductive method is wholly inadequate as an instrument of research. Whatever view may be held as regards the place of psychological implications in a completed body of mathematical doctrine, in research the psychological factor is of paramount importance. The slightest acquaintance with the history of mathematics establishes the fact that discoveries have seldom, or never, been made by purely deductive processes. The results are thrown into a purely deductive form after, and often long after, their discovery. In many cases the purely deductive form, in the full sense, is quite modern."

If pure deduction is barren in the hands of mathematical discoverers, it is not likely that schoolboys will find the process fertile. A schoolboy should be a kind of discoverer. It is for the teacher to put him in such a posture that he cannot fail to make discoveries. But he will fail if he is blindfolded; and this is the effect of discouraging induction and observation.

"Geometry and mechanics are both subjects with two sides: on the one side, the observational, they are physical sciences; on the other side, the abstract and deductive, they are branches of Pure Mathematics. The older traditional treatment of these subjects has been of a mixed character, in which deduction and induction occurred side by side throughout, but far too much stress was laid upon the deductive side, especially in the earlier stages of instruction. It is the proportion of the two elements in the mixture that has been altered by the changed methods of instruction of the newer school of teachers. In the earliest teaching of the subjects they should, I believe, be treated wholly as observational studies. At a later stage a mixed treatment must be employed, observation and deduction going hand in hand, more stress being, however, laid on the observational side than was formerly customary. This mixed treatment leaves much opening for variety of method; its character must depend to a large extent on the age and general mental develop-

ment of the pupils; it should allow free scope for the individual methods of various teachers as suggested to those teachers by experience*."

The arguments used above are not to be forced with the sense of an attempt to drive deductive methods out of mathematical teaching. But they do suggest that deductive methods have tended to usurp more than their proper place. The revolt against this usurpation may have carried some of us too far in the opposite direction, and no one can fail to see the danger than an exaggeration of the experimental and intuitional element may leave mathematics invertebrate. The truth is, of course, that experiment and intuition should predominate in the early stages of a study and deduction in the later. And here again we have to maintain a balanced view. Quite early a deductive element may enter, but incidentally and informally. We may ascertain by experiment or intuition the equality of vertically opposite angles, and of corresponding angles in the case of parallels; but we need not hesitate to deduce the equality of alternate angles. At the other end of the course the deductive flavour will be much stronger; but need we bar experiment and intuition? According to Hobson, this would leave us at a standstill. Take the case of an advanced student reading Theory of Functions. He will find the subject set out beautifully in a deductive form by various writers, but he must be a dull fellow if he is content to follow the deductive guiding thread submissively. It is only the student of the pure examination type who will forego the pleasure of experimenting, conjecturing, "analogising," verifying, ahead of his reading. I submit that this is the actual method of reading adopted by real students; and that the instruction of advanced classes must conform to these actual conditions of thought, or fail. The master must make his pupils stand on tiptoe to pick the fruit; when they have picked it they may digest it deductively at their leisure.

* Hobson, *loc. cit.*

Each new subject that is taken up, even each new chapter, will be approached in an experimental or intuitional attitude, subject always to the dictates of common sense. There have been many advocates in recent years of the opposite plan; all experiment at first, all theory later. This I am sure is a mistaken view; it springs from a deep-seated if undefined belief that mathematical thought is solely deductive, and that any other element in mathematical teaching is a rather disreputable intrusion, inevitable perhaps, but a thing to be got over and done with as early as may be, like measles and mumps. What is really needed is a sensible blend at each stage, and anyone will come to this conclusion who will take the trouble to examine the workings of his own mind when a new study is undertaken.

Experimental and intuitional methods are not identical. I cannot pretend to give a psychological account of the matter, but the distinction is fairly obvious though not always realised clearly. Take the equality of vertically opposite angles. If I measure the angles I am proceeding experimentally; if I open out two sticks crossed in the form of a X, and say that it is obvious to me that the amount of opening is equal on the two sides, then I am using intuition. Intuitive perception of such geometrical truths is perhaps the result of the countless unconscious experiments with matter that we make at each moment of our lives. Many teachers accept intuitional reasonings but are doubtful about experimental work in easy cases. It may be admitted that when a truth is quite obvious to intuition, it is tiresome to pretend to discover it or even to verify it by measurement. But the practical question is, What is obvious to the immature mind? It is not obvious to all boys of 10 that the alternate angles of the letter Z are equal: their intuition has to be cultivated by experiment before this becomes obvious. On the other hand, there is a feeling of make-believe when a class is told to ascertain by measurement that two sides of a triangle are together greater than the third. They

have an intuitive knowledge of this fact, a fact known "even to an ass" as the Epicureans were wont to point out. It is true that many boys, confronted with the formal enunciation of the fact, would fail to recognise it as a piece of their existing knowledge; but this is because they are not yet accustomed to think in geometrical terms. They would recognise the fact practically in the playground, and would then be able, if invited, to translate it into geometrical terms.

One is tempted to give the following as an instance of intuitive knowledge not needing experimental verification—namely, that the diagonals of a rectangle are equal. Most boys would see this if the figure were placed before them symmetrically: a few would not see it even then; and a few would see it however the figure were placed. I should expect almost every boy to recognise the fact in the case of a rectangular field if they were asked which diagonal would take the longer time to traverse. It would be an interesting subject for experiment with boys of 10 or 11.

The following experiment was made in a class of 13-year-old boys at Osborne—neither the best nor the worst of their term. Having been familiarised with the word "rhombus," they were asked whether or no the diagonals of a rhombus bisect the angles. They were given some minutes to decide; and at the end of this time half of them said "Yes" and half said "No."

I have chosen these instances as typical of a large class of properties that may be considered obvious from the intuitive perception of symmetry. And the remark I want to make is this: a practical difficulty is that boys perceive intuitively relations which as a matter of fact are untrue. For every 100 boys who would tell you that the diagonals of a parallelogram bisect one another, there are probably 90 who would say that the diagonals bisect the angles. There is hardly a boy who would not assert that the bisector of an angle of a triangle bisects the opposite side. Boys are always jumping to conclusions. This is

not a phenomenon that need worry us, any more than the phenomena that they have good appetites, or make mistakes in arithmetic. It is proper to their age. If the schoolmaster had no cause to grumble at what he calls the stupidity of boys, then his occupation would be gone.

It is a favourite assertion of strict Euclideans that this habit of jumping to conclusions is not corrected, is even perhaps fostered, by what they call the loose modern way of teaching geometry. Sometimes they say that studying science encourages the pernicious habit. Certainly the old Euclid drill made boys hesitate to jump to conclusions; and, for the matter of that, to jump at all. This negative habit of mind is not favourable to progress. In order to advance, whether as a learner or as an investigator, it is necessary to jump to conclusions, and of course it is equally necessary to check the conclusions. How are we to encourage the habit of checking conclusions without forming a timid habit of mind that will not take one step off the ground? One way is to insist on the advantage of checking results, whether by experiment or measurement or numerical instances or deductive reasoning. This kind of caution is not akin to timidity: it does not restrain from any method of advance that the mind may devise. It is an element of the process of induction.

The word "induction" is often used by mathematicians in a particular sense. It is sometimes possible to show that if a certain identity is true for a value n, then it follows for the next value $n + 1$. But it is known to be true if $n = 1$; therefore it is true for $n = 2$; therefore for $n = 3$: and so for all integral values of n. This particular process is commonly known as "mathematical induction"; and it is referred to here to avoid confusion, for the word induction is not commonly used in this sense in educational discussions, or in the present chapter.

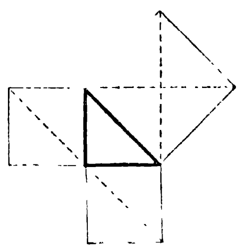
Induction involves these processes: (1) the collection of a number of apparently disconnected data, e.g. the sum of the

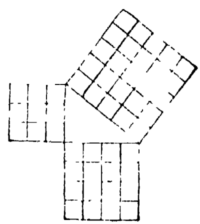
interior angles for a number of polygons of 3, 4, 5, 6, etc., sides; (2) the trial of various hypotheses to harmonise these results; (3) the selection of a hypothesis that seems to succeed; (4) the testing of this hypothesis by some means or other, deductive reasoning being one possible way of testing. The process of induction is continually used by rational beings in everyday affairs; this will be realised by a householder who has to deal with water coming through the ceiling, or an escape of gas. Uneducated people are apt to omit the last step of the process, the testing of their hypothesis; the habit of testing hypotheses seems to be one of those habits that education can cultivate. Let us therefore cultivate it in teaching mathematics. All that we have to do is to form the habit of saying "Have I verified this hypothesis?" Perhaps this good habit will have a better chance of spreading from mathematical to general activities if it is pointed out explicitly, to boys of suitable age, that they are not to leave behind them in the class-room any good habits that they may have acquired there; it is not good to conceal from boys that their work at school has some bearing on their after life, and it might be well if many teachers kept this point more constantly in view.

Intuition enters into the inductive process mainly at the second stage—the framing of likely hypotheses. This calls for sagacity, for a trained sense of fitness, for the mathematical instinct that is latent in most minds. The teacher who begins by enunciating the proposition gives no opening for this mental process, a process from which great pleasure is derived when it is successful. He may allow intuition in the search for a proof; but it is better to search for a proposition first. In life we have to find the propositions as well as prove them.

The schoolboy's induction is a modest affair; he does not, as a matter of fact, marshall a great array of facts and then proceed to form hypotheses. What he generally does is to form a general hypothesis from a single particular fact, a dangerous proceeding for a man of science, but less dangerous for a boy

with a teacher to guide him; I am not inclined to the extreme "heuristic" theory that the teacher should efface himself. Take the case of Pythagoras' theorem. You might make a boy measure the sides of 100 different right-angled triangles; but he might stare at the results for ever without evolving any hypothesis likely to harmonise them. If the master now directs him to square and add, he will probably discover the theorem; and this is one stage better than the old way of leading off with the enunciation. The result may interest, but the whole proceeding is too arbitrary; there is no intuition in it. Now if we begin with a picture like the first figure and follow it up with the second, most boys will induce the general theorem.





Induction is equally necessary in arithmetic and algebra. A formal, scientific discussion of these laws is quite out of place in elementary work; such discussions arose late in the history of mathematical thought, and should enter late in educational progress. For purposes of introductory teaching, a rough induction from a few numerical instances is sound pedagogics.

To sum up, then, the following principles of mathematical teaching have been suggested: (1) deduction is a process peculiarly appropriate to a final statement of mathematical results, but it is not suited to the exploration of new fields; (2) for the latter purpose induction is necessary, and this method should be used also in presenting new matter to a class; (3) induction must be aided by intuition and experiment, but these processes are not identical, and each has its proper place.

CHAPTER IV

MODE OF TEACHING

"Mode" is a convenient word to denote the way in which a teacher spends the hour; whether he lectures, or hears a prepared lesson, or walks round looking over the work as it is being done, or reads the daily paper (a mode with which I was familiar in my schooldays). The word "method" will be taken to refer to the way in which he arranges the subject-matter.

By way of contrast to the English mode, I will first quote descriptions of the modes prevalent in the United States, and in Prussia.

The characteristic mode in the States appears to be the "recitation" system, though in recent years there seems to have been a departure from this system. "Prior to this time (1880–90) there had been a fixity of aim and a definiteness of character in the methods of instruction. The prevailing mode of instruction was the recitation system, interspersed with a few lectures. For study a text-book was used on which definite lessons were doled out daily, and upon these lessons the student was compelled to study and recite." The boys, then, had to get up book-work from the text-book, out of school, and reproduce it in school; very much like a construing lesson in Latin or Greek. This recitation system is not used to any extent in England as far as I know, at any rate in elementary mathematics; the average boy would not make much of the book-work of most English text-books, which generally seem to be written for the instruction of the master.

The Prussian mode of conducting a mathematical lesson is described by Professor J. W. A. Young in *The Teaching of Mathematics in the Higher Schools of Prussia*, a book that every teacher of mathematics should read. "The first thing which

impressed one in the class-work, that which remains finally the most prominent characteristic, was that the *teacher teaches*. He does not 'hear recitations'; he does not examine the pupils to see whether or not they have learned some assigned matter from a book; this custom seems happily quite a thing of the past here. At times he imparts new knowledge himself, especially by way of definition and introductory work, but most frequently he leads the pupils on by skilful questions themselves to discover new truths. In the development of new propositions the teacher guides the work, but the pupils suggest step by step what is to be done next. Home-work and the study of books are very minor features; by far the heaviest stress is laid on the class-exercise....If I were to describe the method of instruction taken as a whole by a single phrase, I should say it is 'the Socratic method,' the method of skilful questioning, of leading the class on to the desired goal by a series of questions, each usually fairly easy to answer in itself....Every bit of the hour's work is vitalised by the teacher; there is not a minute when his voice is not heard, and there is also not a minute when his voice only is heard."

Perhaps we may say that the normal mode of conducting a lesson in English secondary schools was, and to a less extent still is, to make the boys work out individually rows of exercises from the text-book. The good old-fashioned way of beginning a lesson was "Open your book at Exercise 456 and go straight on," no fuss about preparation, presentation, apperception and all the rest of it. New matter was usually explained by the master; I should say that only lazy teachers expected boys to master new matter from the text-books, though I have a distinct recollection of a baffling proof of H.C.F. which I was left to learn for myself.

I suppose that there is a movement towards international uniformity in teaching; it is said that the modern American mode has approached to the Prussian model; and Young's

description of a lesson in Prussia is not altogether inapplicable to what goes on in England now. But my impression is that in the vast majority of class rooms (in secondary schools) considerably more than half the time is spent by the boys in working independently exercises from the text-book or the blackboard—the master circulating among them, giving help and correction, occasionally collecting the attention of the class while he deals with some common difficulty. In so far as this is the mode of teaching adopted, we must recognise it as individual teaching rather than class-teaching: and I am inclined to believe that the characteristic English mode may be described as individual teaching supplemented by class-teaching, both being directed to the solution of exercises rather than to the presentation of book-work.

The adoption of this mode of teaching arises from the view that until a boy can apply a piece of knowledge to the solution of an exercise, the knowledge is not his; that a carefully graduated set of exercises will present the matter to him in a variety of ways and give him an all round grasp of it; that this mode enables each boy to work at the pace proper to his ability, whereas class teaching reduces the best to the pace of the worst; that in this way is most completely attained that unique virtue of mathematical work, its power of demanding effort; that in class-teaching it is impossible to be sure that some boys are not idling under cover of an attentive and interested mien.

There is force in the above arguments, and I have no doubt that individual teaching is necessary. But of course there is another side to the question.

The individual mode presupposes a small class. When the numbers are beyond 16 or so, the teacher cannot go round rapidly enough to correct the work efficiently. In class-teaching each boy gets the full advantage of the master's instruction; in a class of n boys working individually there is

$\frac{1}{n}$ of a master to each boy. The stimulus and inspiration of the master's personality should count for something, and this has little scope when the work is individual; over-teaching is to be avoided, but so is ennui. Some of the arguments cited above take for granted that the master is weak at class-teaching; and a weak teacher may wisely confine himself to the individual mode. Class-teaching is the only obvious way of making a mathematical lesson serve as a lesson in English; clear thinking and clear expression go together, and there is not much room for expression in working out exercises on paper.

My own feeling is that, while retaining the predominance of the individual mode, we may usefully make some further approach to the Prussian mode.

The "attention" difficulty should not be urged against class-teaching. A well-disposed class—i.e. a class in an efficient school—is satisfied if the teaching is *fairly* interesting: they do not call for the standard of a Royal Institution lecture. If the master undertakes to do most of the talking, then he must come prepared to be very interesting; and even then he may not do so well as a less interesting person who talks less. Given the bedrock essentials—discipline, a good syllabus, adequate preparation in previous classes—the attitude of an average class in a mathematical lesson is likely to be good enough. There may be one or two *mauvais sujets* inclined to dream; but it is an essential of English education that the weakest shall go to the wall: if class-teaching is good for the majority, it must not be abandoned for the sake of the unfit.

One way of securing the attention of each boy is occasionally to arrange a lesson in the form of a sandwich; select some problem that lends itself to discussion, have a preliminary conversation (not lecture) about it, then set the class to work some portion on paper; after a few minutes make them put their pens down and report progress; further conversation, and then some

more writing. In America it is usual to make the boys work at blackboards round the walls of the room, an arrangement that lends itself to the sandwich system, for the master can see at a glance how things are going. This wealth of blackboard is not usually found in English schools, but frequently a boy is called out and made to work on the board for the instruction of the rest, the master becoming chairman. Devices of this kind work well when the novelty has worn off and the boys have learned not to hold the chalk like a pen; it is good to harden them to standing up and addressing their fellows, and perhaps this is as useful a lesson as they are likely to learn at school.

An argument used against class-teaching is that boys brought up on this system are unable to use books. To an extent the same remark applies to the individual mode, for here the book is commonly used merely as a collection of exercises, and this kind of work cannot be said to give a boy the habit of getting instruction from a book. It is not till he begins to specialise in mathematics that he is required to master book-work from the book, and most people are agreed that at this stage class-work must begin to give way to individual work.

If it is desired that the habit of really using a book shall be formed early, a specially constructed book must be chosen. In one type of text-book, the matter is set out in final, systematic form, as if the writer was showing how the work should be written out in an examination; as a rule, no attempt is made to lead the reader to anticipate the law, or to exhibit the matter to him as bearing upon any practical situation. This type of text-book may be called the scientific type; as a substitute for class-teaching, or as a means of training a boy to use a book, this type is useless; on the other hand, it is the most convenient type for use in revising book-work for examination.

A book that is to take the teacher's place in presenting new matter must be on quite other lines. The order of presentation must be psychological rather than scientific; the deductive

style of the scientific treatise must be freely supplemented by suggestions for inductive reasoning. The whole matter in fact must be suggested rather than put down in black and white; corollaries and easy propositions must be left proofless, in order that the pupil may have the opportunity of discovering the proof for himself. A book of this kind is not useful for examination revision, and this consideration is responsible for varying degrees of compromise between the two types of book. But any compromise adds to the difficulty of replacing a teacher by a book: and it is very doubtful to me whether mathematics is a subject that lends itself, in the early stages, to the formation of a book-habit.

An American mode known as the "quiz" is worthy of remark: it appears to apply to advanced work.

"In the quiz a review is made of the contents of a group of lectures recently given, and this is done by discussion, between student and teacher, and by sharp cross-questioning on the part of the latter. Not infrequently new material is added in this manner to the lectures already given. The advantage of the quiz is not alone to the student. The teacher is enabled thereby to keep in touch with him; and from the democratic American point of view 'keeping in touch' is a cardinal element of sound teaching. The indefinite spinning of lectures by the teacher, careless whether they are being followed by the student, is foreign to our conception of education. Much could of course be said for the lecture system on the principle 'The devil take the hindmost'; for, undoubtedly, by sifting the students and casting out the weak, the best talent can be most rapidly developed. But one of the characteristic American aims is the development of an intellectual democracy rather than an intellectual aristocracy. It is greatly to be regretted that the quiz is not more frequently used, and its development more carefully studied. In the hands of the skilful teacher a quiz, say once in every four or five lectures, can be employed to

instruct and aid simultaneously the weaker and the stronger pupils. Great insistence can be laid upon accuracy, clearness, conciseness, upon thorough comprehension and expression of ideas; and the slipshod work, due so often to a mere hearing of lectures, can thereby be checked. While the quiz may impede rapidity of progress in a given subject, the loss is more than compensated by the hearty co-operation and understanding which it secures between teacher and student, and by the added interest in the work."

A mode of teaching that has found its way into mathematics of late years is the laboratory mode. This term may be applied, not only to such work as hydrostatics, mechanics, etc., as is commonly carried on in a laboratory, but also to practical heuristic work in geometry and arithmetic for which no laboratory is needed. The essence of laboratory work is that the teacher fades into the background, and the problems are allowed to present *themselves* in a concrete shape. In class-teaching the teacher occupies the stage, even though the class is not permitted to be mere audience: to use a less objectionable metaphor, the teacher is pilot. In laboratory work the teacher only intervenes to pull the cart out of a rut. To a certain extent, laboratory work imitates the conditions of real life; problems in real life are not presented to us in a clear-cut form by an individual or a book; they just turn up, with fuzzy outlines, and we have to define them for ourselves before we can begin to solve them. In practical affairs to define the problem is more than half the battle. The aim of text-books and teachers is generally to put things so clearly as to deprive the pupil of the opportunity of defining his problem.

An experienced teacher may controvert this view of laboratory work on the ground that it is found just as desirable to present a clear problem to a boy in the laboratory as in the class-room. An average boy is very unlikely to discover Archimedes' principles or the existence of π unless his steps

are guided in the right direction. This is perfectly true; the problems of which I have spoken as presenting *themselves* are not the big principles which figure in the schedules. The problems that present themselves, unexpectedly, are the little wayside problems. The ruler is not long enough to measure this line: the paper turns out to be too small for this figure: the metre rule cannot be brought into contact with the barometer tube without some rearrangement of the apparatus: numberless little situations of this kind arise, and call for resource or dexterity. Practical work is bound to be richer in opportunities than any book-work. Things can go wrong in great variety of ways; no teacher could interpose such a multitude of small surmountable obstacles: even if he could, he would get no thanks for his perversity, whereas the perversity of *things* arouses no resentment, but on the contrary stimulates to effort.

CHAPTER V

OUTLOOK IN MATHEMATICS

Aristocratic education means education designed to meet the needs of the ablest boys. This is the system on which a good proportion of present-day teachers were trained. The method adopted was that of wholesome neglect, and a very good method this is when the subject is a clever boy and the teacher a real master: those of us who were pupils of Mr R. Levett at Birmingham appreciate the self-restraint with which he taught his scholarship boys, and the independence fostered by this policy.

The aristocratic theory of education has been responsible for the choice of subject-matter in mathematical teaching, and this is where we are beginning to depart from it nowadays. In the past—the not distant past—it was tacitly assumed that mathematics could appeal only to the few; that the average boy was essentially stupid and more or less a hopeless problem. This entirely false assumption arose from the aristocratic theory; the course was designed for the best boys, and with the object of turning out mathematical scholars. In this, of course, it was bound to be successful as the number of scholarships available was predetermined; whether or no it was equally successful in turning out mathematicians is another question. What it failed to do was to train up a generation of men capable of thinking in a mathematical way and of understanding the relation of mathematics to modern life. It is not surprising that these objects were not attained. In geometry the matter and methods of instruction were purely Greek, and the material side of Greek civilisation was based not on mathematics but on slavery. Greek civilisation depended on forced labour; our civilisation depends on the forces of nature which need more refined methods of management.

When we speak of Hellas we have in mind the cultured few, and to them mathematics meant a good deal, but solely as an intellectual luxury. It may be conjectured that mathematics will never be the favourite intellectual luxury of the educated Britisher; if the subject is to appeal to him, the appeal must come from another side. And here I do not want to be suspected of maintaining that mathematics can appeal to a Briton only as a means of filling his pocket. It is difficult to turn mathematics into much money; a distinguished engineer has told us that he can buy all the mathematics he needs at a very moderate number of shillings per week. Mathematics is not a bread-and-butter subject except for those who are satisfied with this simple diet. The argument must be put on a higher plane.

In England we have a ruling class whose interests are sporting, athletic and literary. They do not know, or if they know do not realise, that this western civilisation on which they are parasitic is based on applied mathematics. This defect will lead to difficulties, it is curable and the place for curing it is school. The study of science in public schools will do much to put this right; but science has not the privileged position that mathematics enjoys; it has not the same opportunities. Mathematics was a well-established subject of instruction in public schools before science was heard of. For generations English boys have learnt mathematics, but the subject has not been taught in such a way as to cultivate a mathematical outlook on the world. It has not justified its privileged position. Partly this is due to the aristocratic theory of education, we have to break this down and think of the average boy, the clever boy will take care of himself, and there is no fear of his interests being forgotten so long as scholarships are the most powerful lever in education. The aristocratic theory has been the bane of education in all subjects; the specialist teacher wants to make a man in his own image; the non-specialist, who might be expected to sympathise with average wits, lacks either the courage or

the originality to strike out a line of his own. It is to the specialists that we must look for improvement. They must be persuaded that it is not necessarily deplorable for their pupils to remain ignorant of dodges and bits of knowledge that they themselves cherish from long familiarity.

But, apart from the aristocratic theory, the main reason why the average boy leaves school with no mathematical outlook is that this has not been accepted as a main object of mathematical instruction. Why not? There is no glory to be won from slaying the slain, and I am tempted to remain silent on the subject of examinations. Examination grinding has been attacked so often and defended so seldom that in theory it should be quite slain; but I have an impression that most of the testimonials I have read recently contain reference to Mr A's success in preparing his pupils for examination; I doubt whether things have changed perceptibly for the better. Most teachers still regard a good examination average as the most real test of success, and the "mathematical outlook" developed under these circumstances is likely to be a sharp outlook for a probable question.

This will continue to be the dominant feature until the time comes when a teacher is trusted to examine his own class. In the meantime there is a minority who sow for a harvest to be reaped elsewhere than in the examination room. What are their ideals?

A section—perhaps the larger section—are victims of a psychological theory. The theory is that in the class-room you can develop certain powers or faculties that admit of being carried over and transferred to the activities of real life. Some hold that the memory can be so trained and developed, though it is fair to add that memory training is commonly attempted through the medium of literary rather than of mathematical studies; respectable teachers of mathematics fight shy of much memory work. However this may be, it is probably impossible

to increase the general retentiveness of the memory by any kind of exercise; the memory is not analogous to a muscle. To come nearer to mathematical practice, it is almost universally assumed that the study of the Euclidean and other derived systems of formal geometry cultivate the so-called "logical faculty." Such studies do no doubt attain their object when the subject-matter is geometry; by studying formal geometry I became more wary in approaching a geometrical argument, and more skilful in detecting fallacies in such an argument.

But do I become more logical in the reasonings of actual life? I should like to answer in the affirmative, but some evidence ought to be forthcoming before this answer is given. I should like to see evidence that mathematicians actually are more logical beings than their fellow men (apart altogether from the question of whether it is well to be logical) I should also like to see a psychological discussion of this logical business; it might be found that the word logical is used in more senses than one. If a trained geometrician is expert in putting his finger on the fallacies in a mathematical argument, is not this mainly because he knows the kind of thing he has to look out for? The fallacy may arise from a bad figure; or from the neglect of an alternative root to an equation or from assuming that an infinite series must converge, and so forth: all these are old friends. I do not know that a mathematician would be quicker than another man to unveil the fallacies of a free trade or protectionist argument, unless the argument were of a mathematical nature; the weaknesses of political arguments often arise from erroneous historical premises. If a training in geometry were of palpable value in fortifying the logical faculty (assuming that such a thing exists) it ought to be easy to find definite instances in which it might be said "So-and-so would not have made this mistake if he had learnt Euclid." But is it easy?

We are very much in the dark about these questions: I should

be unwilling to go to the stake in defence of the statement that Euclid makes a man logical. Even if mathematicians do exhibit certain well-marked characteristics in their habits of thought and behaviour, we have at least two hypotheses before us; the characteristics *may* be due to their course of study, or on the other hand these *may* be innate characteristics which led their possessors to choose the study of mathematics. Even if we adopt the former hypothesis, we have to remember that the fully fledged mathematician has made a very intense and concentrated study of his subject. He has probably studied nothing else for at least 3 years at the University and very likely for 2 years before; and he may have been engaged in mathematical work and teaching for an indefinite number of years before he comes under our observation. An ordeal of this duration does undoubtedly tend to colour the whole mental and moral character. But what we are concerned to ask is rather this—Does a comparatively superficial study of mathematics (or any other subject) such as may be made by a schoolboy not specially gifted in this direction, does such a study as this affect seriously his general mental qualities? Such a study can do much for a boy. It can give him definite knowledge, it can give him interests, it may give him ideas and modes of thought about form and quantity that will shape his way of looking at all sorts of subjects; it may give him outlook, and intelligent appreciation of things going on round about him. So much I think we may make fairly sure of if we go to work in the right way, and later on I wish to make some suggestions as to how these ends may be attained. But when it is said that mathematics develops the memory, the logical and reasoning faculty, the power of generalisation, develops all these powers as applied not only to mathematics but also to general activities—well, I hope that it may all be true, but I have not met with a proof. And unless these developed powers can be carried over from mathematics into general activities, the effect is of no significance in the

education of the average boy; he is not being trained for a mathematical career.

Boys certainly are apt to leave behind them in school any good habits that they may have formed there. They go even further and leave their habits in the particular class room where these were engendered. Cases of this kind are typical: A boy who is practising decimals in mathematics is found unable to divide by 1000 in the laboratory, he may be studying cylinders in mathematics but breaks down over the sectional area of a cylinder in the workshops, a senior class are bowled out by their classical master over a question of sesterces. These cases are too familiar to be a matter of surprise to anyone who supervises the work of more than one department; but to the specialist they are a continual offence; the only remedy is to have a good splice between different departments, so that what is learnt in one class room may be applied in another. A mathematical question set in a physics class-room takes a boy unawares; it comes in a strange form; it is an unusual stimulus and does not automatically provoke the expected reaction. We want to make a boy's knowledge and acquired habits responsive to all sorts of stimuli; hence the need for that system so dear to educationists and such a bugbear to practical teachers—correlation.

The point of this illustration is, that if it is so difficult to "carry over" mental habits from one department to another within a school, how much more difficult is it to "carry over" from school into after life? Perhaps this object would be better attained (if it is attainable) by providing for it more deliberately. For instance, the study of geometry ought to teach that it is necessary to verify hypotheses. The framing of hypotheses depends on a cultivated power of guessing, a kind of *flair*. This guessing is a necessary part of mathematical thought, but the guess has to be verified in some way or other, in geometry generally by deductive reasoning. In solving a geometry rider, I may guess or suspect or have an intuition that a certain pair

of angles are equal; I may be unable to solve the rider till I "spot" this, and the guess may be half the battle. The other half consists in verifying the hypothesis. Now it is of no avail that a boy should be habituated to verify his hypotheses in geometry unless he is able to "carry over" the habit into real life; the commonest cause of vulgar errors is the failure to verify hypotheses. But it is very doubtful if the habit is often carried over, and I think that good work would be done if boys were explicitly taught that this habit is formed for them in school in order that they apply it out of school; probably very few would realise this unaided.

Whether this faculty-training view of mathematics is right or wrong, it has held the field for some generations, and has had every chance of showing what it can do; what are the results? A referendum would say a failure. Whether or no we have done all that is possible we have certainly failed in one thing; broadly speaking, we have failed to make mathematical thought enter as a main element into the life of the educated classes. To redeem this failure is work for the present generation of teachers, and I suggest that we shall succeed if we think less of the faculty-training and more of the "outlook" aspect of mathematical teaching.

What is there in the present syllabus to give boys a mathematical outlook? For the average boy the syllabus comprises arithmetic, algebra and geometry and nothing else. My thesis is that the treatment of these subjects should be so remodelled as to leave time for a further range of subjects and a wider field of ideas. Considering the time given to the subject, the amount of mathematics covered by English schoolboys is insignificant. To the faculty trainer this is a matter of indifference, as the training can be given as well through a thin syllabus as through any other. From the outlook point of view it is not a matter of indifference. More and more the affairs of life are being made amenable to mathematical treatment, and as it has turned out

the development has been on lines divergent from the lines of school-work. The form of mathematics that nowadays is inserting itself into so many departments of thought is the infinitesimal calculus. For general purposes the technicalities and machinery of the calculus are not needed, but the language, notation and ideas of the calculus are of all-pervading utility. Leaving aside the definitely mathematical sciences of physics and engineering, we find the calculus entering into such studies as chemistry, biology, economics and statistics. Lectures on these subjects are apt to be unintelligible to an audience brought up on school mathematics, for want of a nodding acquaintance with the calculus; at some universities it is found desirable to arrange special mathematical lectures in order that students may be able to follow the instruction in other subjects; a kind of death-bed repentance for those who have wasted their mathematical time at school through no fault of their own. And let it be noted that this simple form of calculus does not grow out of the summit of school mathematics, but branches off low down the stem; it is independent of formal geometry, and a vigorous pruning of school algebra and arithmetic would in no wise prejudice the growth we want to encourage.

The infinitesimal calculus has now been before the world for $2\frac{1}{2}$ centuries. It is the fundamental form in which mathematics are applied to the affairs of modern life. We must recognise as a law of development in educational affairs that matter which in one century occupies the attention of the foremost philosophers finds its way in a subsequent century into the elementary curriculum. If the infinitesimal calculus was the high-water mark for the 17th century, so was the Euclid's geometry to the 3rd century B.C., the Arabic notation (in Europe) for the 12th century A.D., the method of long division for the 15th, and so forth. The quality of the human brain does not alter, presumably, from one century to another; how then has it been possible to make the schoolboy of one generation assimilate

matter that has puzzled the best brains of earlier times? The answer is simply this: teachers come to know more, simpler methods of presentation are discovered, and a clearer view is attained of how the curriculum can be disencumbered of the obsolete and the unessential. No one who has taken the trouble to acquaint himself with the world-movement in education will doubt that one of the tasks of the 20th century is to find a way of importing the notions of the infinitesimal calculus into the ordinary school curriculum.

What has the ordinary syllabus to give a boy in the way of outlook?

The world has moved on and left school mathematics in a backwater. Geometry stands a venerable monument of antiquity, on which I will lay no sacrilegious hands. But mathematics is applied to modern life in an analytical or algebraical, not in a geometrical form. Newton revolutionised scientific thought with a geometrical treatise, but he had arrived at his results by analytical means, inventing the calculus for the purpose; and his successors have not thought it necessary to clothe their work in a garb which respectability in those days demanded. Formal and demonstrative geometry is not going to help us very much on the side of outlook; it must be taught as mental training, for we can hardly break with the training theory though our hold on it may be weakening.

Now consider school arithmetic: what does it contain? It teaches in the first place ordinary cyphering, a necessary art. It teaches all sorts of operations with English weights and measures, and coinage; mostly superfluous. It betrays its commercial origin by treating of a number of commercial rules which may have been practised in the City in Cocker's day. But mainly it is a complete collection of methods for solving all problems that have been set by all examiners since the invention of printing, a snowball that still grows, a burden to boyhood, a nightmare to mathematicians. "If all of this were

cleared away," they said, "it would be grand." If it were cleared away, we might discern the true simplicity of arithmetic. The invention of logarithms has left much of arithmetic on the scrap-heap, a shrine where it is still worshipped. There is no outlook to be got from arithmetic; the main thing is to have done with it rapidly, and ever after to use it as a tool.

Algebra is perhaps the mathematical subject which gives the smallest return for the time spent on it, and I shall indicate the cause of this.

It is, I am afraid, a fact that in this country the main preoccupation of teachers is to impart to their pupils a high degree of mechanical manipulative dexterity in handling algebraical expressions. Now this statement does not imply that teachers try to make boys manipulate algebraical expressions without understanding what they are doing. I am not speaking of ordinary bad teaching; but I am saying that a great number of very competent teachers are inspired by a wrong ideal. They want boys *to understand in order to manipulate correctly*, whereas their ideal should be just reversed. The ultimate aim should be, not manipulation, but understanding and outlook.

English education is dominated by examinations. Examiners cannot test outlook and they can test understanding only by testing manipulation; teachers have to supply what examiners demand; it is not surprising then that many teachers have mistaken the means for the end A vast amount of time is spent on purely mechanical work; highest common factor, fractions and factors beyond the types needed for practical purposes, needlessly heavy equations; together with all sorts of artifices and elegancies, which are to the average boy as pearls to swine.

Now if a boy is certainly destined for a career in which he will be bound actually to make use of mathematical manipulation, a case might possibly be made out for drilling him at

school to a high degree of dexterity in the technique of algebra; just as a student who aspires to be a professional pianist must devote an astonishing number of hours to the technique of playing the piano. But for the moment, we are considering the case of the general student of mathematics, the non-specialist. We may assume that the average man, not connected with any mathematical or scientific profession, finds practically no occasion in the affairs of life to enter into the details of an algebraical calculation; it is even more certain that, if such an opportunity presents itself to him exceptionally, the opening is declined, in spite of (perhaps because of) the heavy drill that has darkened his schooldays. On the other hand, those of us who believe in a mathematical training are profoundly convinced that such general mathematical ideas and modes of thought as may be wrought into the mind by a suitable course of instruction are an asset of permanent cultural value. It is these ideas and modes of thought that we regard as a necessary element of a liberal education; manipulative dexterity, on the other hand, we look upon as a purely *specialised* technical accomplishment. It is a specialised technical accomplishment just as much as dexterity in glass-blowing, or Latin verses, or machine drawing, or playing the piano, or shorthand; all of them very excellent things in the right place, but by no means essential to a liberal education. If there were no other way of using mathematical time than in giving this technical skill, I should say at once: Cut off some time and give it to English, or Science, or something else of general cultural importance. But naturally I do not take this view; on the other hand, I am persuaded that a drastic abatement of this juggling with algebraical symbols would free enough time to put every ordinary boy in possession of the fundamentals of trigonometry, mechanics and the calculus. Mathematics might gradually become, for the nation at large, a thing of real significance; and we should no longer have headmasters writing

to the *Times* about "the transient but blighting shadow cast over their schooldays."

This question of lightening the algebra syllabus is not new; the Mathematical Association Committee have taken this particular problem into consideration, and at the beginning of the present year* it issued a report containing definite suggestions for immediate action in this sense. The report is entitled, *The Teaching of Elementary Algebra and Numerical Trigonometry* (G. Bell and Sons, Ltd., Portugal Place, Kingsway, 3*d*.). It expressed a hope "that it will be possible to influence the demands of examining bodies in such a way that the teachers will have freedom to put to better use much of the time at present spent on the elaboration of algebra in elementary classes. Many teachers wish for opportunity to develop with their pupils mathematical ideas that they feel to be of greater educative value—ideas drawn from mechanics, mensuration, solid geometry, infinitesimal calculus, and more especially numerical trigonometry. Custom, represented by public examinations, has at present the effect of withholding that opportunity." Throughout the report the dominant demand is "freedom for the teacher to use to better advantage the time at his disposal." There is no attempt to constrain conservative teachers to abandon their well-understood ways; examiners are merely requested to set "questions involving numerical trigonometry and other subjects that it is desirable to introduce... as alternatives to questions now set on parts of the work here considered to be non-essential." Again, "we do not wish to fetter any teacher in his endeavour to provide what he considers the best general education for his pupils; we wish rather to restrict the demands of examiners to things of real moment, in order to give greater freedom to educators who are anxious to progress."

It is interesting then to see what are the principles on which

* This was written in 1911.

the report determines the portions of algebra that are "of real moment." The first principles appear to be as follows:

"That within the range of work selected the teaching should be thorough, so that at each stage a boy may acquire the facility necessary to enable him to pass on to the next stage without being hindered by lack of skill in the preceding manipulations."

This should dispose of the fear that it is intended to frame an algebra course consisting entirely of ideas, a filmy texture with no substantial element for a boy to get his teeth into, a diet like Falstaff's "two pennyworth of bread to such a monstrous deal of sack." This is not the intention at all—"within the range of work selected the teaching is to be thorough." But the work selected should form a series of stepping stones, leading somewhere; not a parade ground for practising the goose-step.

The second principle of limitation is as follows:

"That a boy should not be required to possess more manipulative skill (in algebra) than will enable him to deal with such parts of the subject as for the reasons either detailed above or otherwise generally admitted ought to form part of a liberal education; in other words, manipulative processes should be developed, in the elementary course, just so far as they are really subsidiary to the aims of that course and no further—not as mere curious exercises."

A description in detail of the results deduced from this limitation would probably be out of place in the present chapter; it is enough to indicate that the results are considerable. If algebra teaching were confined to the essentials recommended by the M.A. Committee, a very important change would come over the conditions under which the average schoolboy works. At the age of 12 he has learnt what an equation is, what it is for, and how to solve it if it is easy. At the age of 16 he has

looked into the matter further, and found it not to repay attention: he has decided that x-chasing is not his vocation. Probably he is right and his teachers are wrong: whether they are right or wrong they have to satisfy the examiner, whose demands are stale but persistent.

The adoption of some such policy as the M.A. Committee has put forward might bring about something like a renaissance in English schools. This has actually happened at many schools, where reform on these lines has been gradually evolved; the conventional schoolboy attitude toward mathematics has entirely changed, as is patent to such impartial spectators as classical house masters. The most serious opposition to a general movement this way will come from teachers; examiners generally cede to a fairly universal demand. There are plenty of keen, efficient teachers who cannot persuade themselves that their dear old haunts are blind alleys; year after year they go on trying to make silk purses of sows' ears, trying to turn their sensible, clumsy, ordinary pupils into skilled analysts. They are wasting their energies just as sadly as their colleague who teaches Greek to a middle form. Perhaps they would have more sympathy and insight if they had once been average boys themselves.

If time can be saved from algebra, there is no difficulty in using it. Why should a 3-dimensional boy be tied down to a 2-dimensional geometry? Given the time there is much that could be done to strengthen his space-intuition. There is a movement in France to amalgamate definitely the teaching of plane and solid geometry from the outset, but the particular shape taken by this movement might not suit English needs; it involves a simultaneous *formal* treatment of the two branches, and I doubt whether we ought to be formal in teaching "solid." We used to try this *via* Euclid's 11th book, but the result was poor. But the idea of simultaneous treatment is sound, if difficult to work out. It might take the form of an informal

reference to 3 dimensions, accompanying the formal 2-dimensional course. In this way might be introduced those simple considerations about parallel and perpendicular planes and lines that are so fundamental and yet so little understood in general, and the matter of so much confusion of language in everyday life. There might be an amplification of this after the plane course, with a discussion of particular solids, the use of the globes, and a little descriptive geometry. The mensuration of solids yields to the magic touch of integration.

Now that geometry has become numerical and mathematical tables no longer a luxury, trigonometry cannot be kept out of the non-specialist course. The step is easy from measuring angles on paper to measuring angles with a simple instrument of the theodolite species; there are some who would say that the step should be reversed. But real people do not go about measuring elevations without a purpose, and why should schoolboys? If they take an elevation, let them get something out of it, the height of some object in which they are interested. To do this by drawing teaches first what a good weapon drawing is, and secondly the nature of similar figures; the word "similar" should be heard of here or hereabouts. Now drawing is a method which no one need be too proud to use at any age, but after the earlier stages of secondary school life its use should be merely incidental; where a boy has mathematical tables and has taken elevations, it is very arbitrary to withhold from him the proper way of reckoning the height by the tangent of the angle. Further than this, it is needed by the physics teacher with his inclined planes and tangent galvanometers. Trigonometry therefore refuses to be excluded from the non-specialist course: and I am not one of those who regret that boys find it an easy subject; there is no fear that mathematics will ever be too easy. And with trigonometry, we begin to get the "outlook" which is the text of the present chapter. For a boy can now be brought within sight of a long row of applications to various

matters of general interest and importance. This arises from the fact that the mathematical tables we teach him to use represent a vast source of energy, the stored-up brainwork of former generations, a source of energy too that does not waste as it is used. In learning trigonometry, he is learning how to tap this source.

Trigonometry is liable to just the same educational misuse as algebra. The trigonometry taught in all schools 30 years ago, and no doubt taught in many today, consisted mainly of algebraical manipulation. It had altogether lost sight of the utilitarian motive to which we must appeal if we want to have the natural boy with us rather than against us. If it is to stand in the non-specialist course, trigonometry must avoid all developments except such as have direct application to concrete and geometrical situations; to step beyond this line is to step into the specialist course. The amount of trigonometry indicated may be covered in three lessons a week for two terms.

We now come to mechanics, which is often taught by the physics master. A course of physics is an essential part of a liberal education; it is almost safe to say that this fact is now recognised practically in schools. Mechanics always enters into a physics course unless it has been delegated to the mathematician; the question is not whether or no mechanics shall be taught, but rather who shall teach it.

A joint committee of the Mathematical Association and the Association of Public Schools Science Masters recommended in 1909 that "statics be begun in the lower part of the 'Upper School' as a part of the regular mathematical teaching, i.e., it should be taught by the mathematical master during mathematical hours." It may be assumed that physics masters would like to surrender mechanics to the mathematician, provided they could be assured on one point—How will the course be begun? Unless the various laws of statics are induced from experiment, the course is not likely to be of much service to the

average boy. The difficulty is that many mathematical masters are unfamiliar with experimental methods: but a keen man can easily pick up enough to teach statics. If there is no mathematical master available of the experimental turn of mind, mechanics might be left to the physics master. But this would be the case in few schools, and there should be an increasing supply of men capable of instructing in both mathematics and physics. I am all in favour of amalgamating the mathematics and physics staff. The physics or engineering man, if he is a competent mathematician, will often be a better mathematical teacher than the pure mathematician; the latter will generally be disinclined to regard his own subject as a tool: he may treat it as self-contained, insular, having no foreign policy. The more mathematical teaching looks outwards, the better will it be for schools, the place for self-centred mathematics is the university.

The main reason for turning over to the mathematician the elementary instruction in mechanics is this, that in any case mechanics is taught by the mathematical staff to the top classes in connection with scholarship work. In the past, there have been two separate and unrelated systems of mechanics teaching within each school; the elements taught inductively by the physicist to the middle classes, and the whole subject taught again, deductively, by the mathematician to the top classes. There might be something to be said for this double system if the two courses were deliberately designed to fit in with one another, but the number of schools in which this occurred must be small. I imagine that pretty often the mathematician did not know that any other system of mechanics teaching existed in the school: there is frequently in schools a lack of touch between departments which would do credit to a government office. It is the business of the headmaster to see that such things do not occur; but not many classical headmasters are competent to exercise general supervision over the mathematics and science of their schools, and the ignorance of head-

masters is itself a result of the failure of mathematical teaching in the past.

When mathematicians do take over the whole teaching of mechanics, they will have to struggle consciously against the temptation to turn it into a set of mathematical problems. Mechanics will be valuable to the average boy in so far as it creates in him a vivid perception of laws of physical phenomena; the niceties of mathematical elegance tend to distract his attention from the fundamental principles, and even to cultivate wrong impressions through such abstract conceptions as the perfectly rough insect, the small elephant whose weight may be neglected, and so forth. It is true that this fauna is almost extinct, but we still hear too much of the frictionless machine and too little of the efficiency of actual machines.

The case for statics in the non-specialist course is quite clear; when we come to speak of dynamics, there is much difference of opinion. In these days of electric motors, automobiles, and aeroplanes it would seem that everyone should be allowed to gain clear notions about energy and force; but my own experience is that it is very difficult to give clear notions. Perhaps there is something in the contention that boys are increasingly familiar with machines and power in various forms, that all this is a very modern development, and that what was difficult 10 years ago may be found much easier today. I should not like to deny the force of this, but still I should hesitate to count dynamics in the non-specialist mathematics course. General notions about work and energy should enter into statics, *via* the efficiency of machines; but precise quantitative knowledge concerning mass, force and momentum and kinetic energy have been attained slowly in the growth of civilisation, and appear to be essentially difficult.

If we restrict dynamics to the study of motion—velocity and acceleration—we are easing the burden very appreciably. Ideas of velocity and acceleration are within the range of geometry.

It is an obsolete dictum that motion—that is space, plus time—is excluded from the scope of geometry.

One of the chief tendencies of geometry teaching is to use every opportunity of presenting continuous *change* of configuration. If we may for the occasion use the words statical and dynamical in the popular sense of *at rest* and *in motion*, we may say that the tendency is to make mathematics dynamical instead of statical. Mathematics was statical when it dealt entirely with things *in statu quo quiescendi*. Everything was fixed and immovable: determination of roots of equations, study of fixed geometrical figures. But now we regard algebra as concerned less with the determination of values than with the study of relations between variables. The idea of function is hovering over school mathematics; the graphing of functions has already found admittance, but this was but the first glimpse of a profound change which is coming. We are beginning to realise the meaning of the πάντα ῥεῖ of our schooldays; the Greek philosopher's discovery was brought within the range of mathematics by Newton's fluxions, and now, after 24 centuries, it has filtered down to the schools.

When once it is granted that geometry is to tell us not only how figures stand but also how they move and change, it is a short step to the space-time diagram and the idea of velocity as represented by the gradient in this diagram. This brings us face to face with the infinitesimal calculus.

Text-books have been published in which all the main applications of differentiation and integration are exemplified without using any function more abstruse than x^n. The reader learns how the calculus bears upon velocities and accelerations, maxima and minima, relative errors, definite and indefinite integrals, areas, Simpson's rule, volumes, centres of gravity, moments of inertia, work done in stretching strings and by expanding gases, mean values.

In all this work the manipulation is slight, but the value for mental enlightenment is immense. A store of applications is

thrown open by the very simplest tools that the calculus provides. The ideas, then, of the calculus, and a feeling of the extraordinary power of this new instrument, are accessible to a student with a modest degree of manipulative skill in algebra. It is not necessary to tread for years the weary paths of Highest Common Factor, fractions and the like, before becoming worthy to enter this rich country.

Now it is not proposed that the calculus, in so far as it belongs to the non-specialist course, should cover all of this work. The non-specialist cannot integrate $\frac{1}{x}$, for this involves e. Many of the applications enumerated above will be beyond his range. For all this, there is much that he can do. He will be like the prospector picking up the first nuggets on a new goldfield. Having been trained to think functionally, he will have little difficulty in grasping the idea that the *gradient* at any point of a curve (say $y = x^2$) is a function of x. On a copy of the graph, lithographed on paper ruled in inches and tenths, he may draw tangents with a ruler, and measure off the gradients as ordinates; this will show him that the gradient is a function of x, and probably he can now see what function. After a suitable amount of preparation in this style, he may proceed to determine the gradient analytically. He is now in possession of the root-conception of the calculus, and the next step will be to space-time diagrams and velocities. Next may come the differentiation of x^3 and $\frac{1}{x}$, and further applications; maxima and minima will present no insurmountable difficulties. Finally the integration of such simple powers of x, and the easy applications that stand at the threshold. This will be the coping-stone of the non-specialist course.

That this range of work is within the powers of the average boy at an English secondary school is proved, not only by the testimony of English schools that have made the experiment, but also by the universal experience of the classical sides of French *lycées*.

PART II

GENERAL TEACHING POINTS

BY A. W. SIDDONS

GENERAL TEACHING POINTS*

I have put this first, not because it should be read first, but because I hope the reader will come back to it again and again.

THE IMPORTANCE OF PLANNING A LESSON

R. L. Stevenson says somewhere that on a walking tour one should always have a plan if only for the pleasure of departing from it. To every teacher I would say "always plan out your lesson beforehand, but do not be a slave to your plan"; sometimes you will find it necessary to spend more time than you expected over the early part of the lesson, sometimes a chance question from a child may give you an opportunity to develop some important idea that you had not thought of when planning your lesson; but, in general, do not be led too far from your plan.

To the young teacher, a plan for a lesson is all important: he will find that his grip of the class is all the stronger for having a lesson well thought out, and knowing exactly what he is going on to as each piece of work is finished off.

With the inexperienced teacher, and even with others, how often does one see delay in getting a class to work the moment the lesson should begin. An excellent plan is to start the class on some short sums which have been written on the board before the class come in; it helps them to get into the habit of getting down at once to the job in hand—an excellent training for life—it gets their brains working and so gets them ready for the teaching that is coming later in the lesson; it tends to encourage quickness which is such an asset; also, however badly the rest of the lesson may go, each child has at least made some

* Though I have taught and examined girls and inspected girls' schools, my main experience has been with boys, so I have often drifted into talking of "boys" and "masters" when I should have said "children" and "teachers." I hope mistresses who read this book will forgive me.

mental effort over these short sums. I shall return to this in the next section.

Sometimes one finds a minute or two left at the end of the lesson; this can be used for odd revision, e.g., multiplication tables, odd facts that are useful, $91 = 7 \times 13$, the squares of $1, 2, 3 \ldots 12, 13, 15, 16, 25$, tables, etc. Have a collection of such things ready for use in case such an odd minute occurs.

If only the teacher will set the example of not wasting a minute, his class may develop the habit too.

THE BEGINNING OF A LESSON

I have suggested above that most lessons should begin with some short sums which the class should start on the stroke of time.

If these sums are marked, the master need not worry much about marks for the rest of the period and he can feel free for real teaching (see Marks, p. 64).

It is worth spending some time on thinking out what these sums should be:

(i) There are drill sums on the work of previous terms. These should be set to improve speed and accuracy and to keep the earlier work revised and fresh.

(ii) There are sums that the child can work in its head—mental arithmetic or algebra or geometry.

(iii) There are questions on the work of recent lessons. These should show up which children have got a grasp of the work and which need further explanation.

The time devoted to this work should be strictly limited, five minutes, or ten minutes at the most.

As to marking this work, sometimes the master should take it home and mark it; in that case it should be on some topic that does not bear on the rest of the lesson—it is useless to collect information on a subject and then to have a lesson on

it unless the information collected has been already digested. Sometimes he should read out the answers and let each child mark his own paper or his neighbour's (this can be done very quickly if it is organised well at the beginning of term), sometimes the better members of a class may be turned on to mark all the papers while the master is devoting his time to the weaker members of the class; but in any case the master should at least glance through each child's paper.

"THE NINE QUESTIONS"

A very able preparatory schoolmaster has devised what he calls "the nine questions." The plan has been in use for ten or twelve years, so there must be something in it.

"The nine questions" are given at about alternate lessons; when new ground has been broken they are always set at the next lesson; they are always set at the beginning of a lesson.

The blackboard is divided off by chalk lines into nine rectangles, in each rectangle is one question (see examples later); each boy is provided with a sheet of paper (about 4 inches by $3\frac{1}{4}$ inches) folded into nine rectangles and he writes the answers in these. The questions are always meant to be done mentally, but sometimes the boy's mental ability is over-estimated, then working on the back is allowed.

As soon as boys finish they are allowed in pairs to compare answers and argue quietly about them.

The time taken for "the nine questions" is usually nearly ten minutes, never more.

The number of marks given for each question is equal to the number of boys who fail to give the right answer.

The plan was devised for purposes of marking. Its survival after several years' use is due to two by-products, (i) the quiet argument of boy with boy teaches both parties very effectively, and (ii) the little test of essential points almost invariably brings

home forcibly to the master that his teaching in the previous lesson was much less effective than he thought it was.

On pp. 71-79 will be found some sets of "nine questions" which I happen to have. They show the idea and anyone ought to be able to make up such papers for his own classes.

THE REST OF THE LESSON

It is impossible for all lessons to conform to one plan; in some lessons the class must be set down to do examples on the topics that have been recently discussed, while the master goes round and deals with individual difficulties, or goes through with each boy the mistakes he made in written work that has been looked over out of school; but when new ground is to be broken, the lesson should be roughly divided into three stages, (i) some revision to prepare the class for the new work, (ii) the presentation of the new work, and (iii) the consolidation of it.

First stage. If the short sums or "the nine questions" have been already marked and if they are related to the new topic, a short discussion of some of the mistakes will provide the necessary revision; or perhaps the mistakes in the short sums of the previous lesson (looked over out of school) or other old written work will provide it, or the master may have to provide it without any text.

Second stage. The presentation of new work is the main subject of this book and I will say nothing more of it here except that it is useless to attempt too much at one time.

Third stage. The consolidation of new work is most important. It is of little use to present new ideas unless the class can consolidate them at once by applying them to examples; this may be done by the master working examples on the board with the class doing the vital steps on scrap paper (see pp. 67, 187) before they are put on the board, or by careful written work with the master going round helping where necessary.

A SECOND LESSON ON A NEW TOPIC

In some ways the second lesson on a topic is more difficult than the first; the teacher has to cover the same ground as before without boring some of the class by dwelling on points which they have already grasped.

One thing I would warn him against, and that is expecting the class to pay full attention if he merely repeats the explanation he gave in the previous lesson. Let me take an example. Suppose that in the previous lesson they have learnt how to solve a quadratic equation by the square root method; will it be necessary to explain the method again? Yes, it will be necessary; but will the class pay attention and get the full benefit from the lesson if the master starts by solving another quadratic on the board? A far better course for him to pursue is to give the class a quadratic to solve for themselves and then to work it on the board after they have done their best at it. But why is that a better course? Years ago I should have said that I could not give reasons, but that my teaching instinct made me feel that it was better, and that experiment and observation proved that it was so; but now I think I can see reasons for preferring the course I suggest. First of all, but for the preliminary trial, many of the class would think that they could solve the quadratic, so they would not pay active attention; secondly the boy who had tried and failed would concentrate his attention on the point at which he had failed, he would ask questions about it, in fact his attention would be active instead of passive, and this second lesson would prove much more effective than the other course I mentioned.

THE IMPORTANCE OF DRILL

I have mentioned above the importance of consolidating new topics. This must be done by some concentrated work on examples, followed by occasional examples set perhaps as short

sums at the beginning of subsequent lessons. When once a process has been mastered, if it is to be a real possession, memory needs to be tickled at intervals. An occasional test-paper on the work of the term is an excellent guide to the master, as well as a useful refresher for the class. The work of previous terms should also be revised regularly by occasional examples and test-papers.

Unless a master makes a point of consolidating the new work and of keeping alive the work a boy has done in recent terms, he is not playing the game by the masters who will take the boy later. This must, to a large extent, be left to each master's conscience, but it should be watched so far as possible by the master in charge of the mathematical teaching; in a secondary school or a public school this should be easy, but it is much more difficult where boys pass on from preparatory to public schools.

Some preparatory schoolboys are pushed on much too fast: they may absorb new work readily, but they do not get enough drill at it for it to stick. I realise the temptation for the preparatory schoolmaster, but in the boy's own interest he should eat the solid bread and not merely lick the jam off it. I have known cases, exceptional I grant, of boys who have done permutations and combinations and the binomial theorem and yet made quite elementary mistakes in dealing with fractions and could not solve an ordinary quadratic equation. The result is that at their public schools they have to go over the elementary work again, and it is so long before they have consolidated that that they get bored with mathematics. In general, if a preparatory schoolboy has covered the ground of the schedule given in the *Headmasters' Conference Report*, his mathematical position at his public school will benefit most if he merely keeps that alive and strengthens himself in Latin or some subject in which he is weak; by doing that he is likely to get into a higher form at his public school, and so to get a chance of new

mathematical work there at an earlier stage than would otherwise be possible.

MISTAKES AND CORRECTIONS

When a master has gone carefully through a bundle of work done by a class, he has made many corrections and notes on the work, but his labour will be largely lost unless he drives home by word of mouth some of the points that arise. How is he to do this?

Some of the mistakes will be common to many boys and may be discussed with the whole class, but the master must beware of treating too many mistakes in this way: he should select those that are of fairly general interest—it is very boring to the boys who have not made the mistake and is largely a waste of their time. Other mistakes should be discussed with the individual boy or a small group of boys; and yet other mistakes will need nothing beyond what the master has written on the boy's paper. In going through a boy's work I always write "Ask" against any mistake about which I wish to speak to the boy; I suggest "Ask" with a circle round it for a mistake to be discussed with the whole class and "Ask" without the circle for an individual point.

Years ago it was customary in some schools (particularly in girls' schools) for almost every sum that was done wrongly to be corrected by the pupil or even done again. No doubt there was disciplinary value in this, but was there any educative value?

Mistakes may be either mistakes of principle or mistakes in computation. Mistakes of principle must be explained; I have already dealt with that class of mistake.

Mistakes of computation may be classed either as accidental mistakes or as consistent mistakes, and it is hard to decide to which class any particular mistake belongs; there is nothing much to be done for the accidental mistake, but the consistent

mistake needs some special drill if it can be discovered. Some years ago I found a boy, who was generally very accurate, often made mistakes when an 8 was concerned; five minutes' drill on 8 for two or three successive days cleared out his mistakes; but it is very hard to spot a weakness like that. If a boy is frequently making mistakes of computation, it is a good plan to make him do the computation aloud, the master may then diagnose the trouble and see what individual drill will put him right.

On the whole I do not think there is educative value in having sums done again; sometimes no doubt it is desirable, but in general I think it is better to have a similar sum done, but every case must be judged on its merits. Mistakes in general seem to point either to lack of concentration or to the necessity for drill on some earlier work or on some special step in the work in hand.

MARKS

I hate marks. I often tell a class at the beginning of term that I can either produce accurate marks or teach them a fair amount, but that I cannot do both; I ask them which they will have and they invariably choose to be taught. This does not mean that I do not take a lot of trouble to produce as fair and accurate marks as possible, but it does mean that teaching is my first object and I do not sacrifice that to marks.

Marks are a necessary evil, but at a certain stage with young children they can be a stimulus and the wise teacher will neglect no stimulus; in a way, children regard marks as the reward of their labour and an index of progress. Marks may be used to some extent in this way, but with boys of fourteen and upwards I generally try to give them a soul above marks and to fire them with a desire for power and knowledge.

In spite of all I have said above, a master must produce marks, and a weekly or fortnightly order can be a stimulus even to the class that has a soul above mark-grubbing.

The question I want to discuss now is how to produce the necessary marks.

Is it necessary or desirable to mark all a boy's work? Emphatically "No"; to attempt such a thing would be too deadening for the master and would merely sap his energy for teaching. Perhaps the class has had a first lesson on some new subject of rather a mechanical nature, say the solution of quadratic equations by the square root method; it has been discussed *vivâ voce*, examples have been done on the board and the last quarter of an hour of the lesson has been devoted to written examples; should the master go through all the work and mark it? Sometimes "Yes"; the resulting marks will show which boys are quick at picking up the new idea, but they will not show which boys will be best at this work in a fortnight's time; but the labour of going through every boy's work is rather deadening and, after looking over half a dozen selected boys' work, the capable master will have a good knowledge of the class's grasp of the new work and he will have seen about every possible type of mistake, so that he will be armed for the next lesson on the same subject and he may be well advised not to try to get a set of marks off that work. He will probably get much more trustworthy marks by setting a few carefully selected questions in a day or two's time; then, too, he will probably find that the majority of the class merely want some occasional drill at the subject and he will know the two or three boys who need more individual help; he will make a note of these and watch them specially. (He will probably have guessed from *vivâ voce* work who these two or three are.)

In the last paragraph I have considered what is reasonable in the case of rather mechanical work; but suppose the work was such that the wording was all important, then each boy's work should be gone through carefully and the wording corrected. In such a case the master can mark each exercise α, β, γ or δ and give 4, 3, 2 or 1 mark accordingly—the marks need not

depend on the number of sums done but mainly on the style.

But such marking alone will not produce a fair enough result, and it is desirable once or even twice a week to set a test-paper on the work that has been done since the previous paper; this, of course, must be marked carefully and the marks given considerable weight. Quite apart from the value of the marks obtained by this method, there is great educational value in such a test. The snag of trusting largely to this method of getting reliable marks is that some boy may be absent for the test-paper.

Then there are the marks obtained from the work that was done at the beginning of each lesson.

With these various sets of marks the master has enough material to produce a very fair order, whether it is a weekly, fortnightly or terminal order, and during most of the time in school he can forget marks and think only of teaching, or rather educating the class, and that is his real job.

SHOULD WORK ALWAYS BE DONE IN FINISHED STYLE?

A lot of the slovenly work that one sees in children's ordinary work and in examinations seems to me to be due to teachers not recognising two standards of work. Sometimes a class should be allowed to work in abbreviated style, i.e., to get results as they please, leaving out reasons and explanations but, of course, keeping the work neat and legible; at other times they should work in finished style, giving their reasons and explanations. In examinations they should understand that finished style is always required, but in their ordinary work they should be told which style is expected.

If all work is expected to be done in finished style, it takes up too much time and tends to produce boredom; but worse than that, the child's best style degenerates, and reasons are left out

and explanations reduced too much. If the two styles are adopted, much more ground is covered and the work done in finished style is of a better standard.

THE ABUSE OF SCRAP PAPER

Whatever style is being used, all the child's work should appear on the paper he shows up and it should all be neat. Some boys have a pernicious habit of keeping a piece of scrap paper beside them and doing much of the computation on that. Other boys work out every example first on scrap paper and then copy it out on the paper to be shown up. This laudable, though mistaken, striving for neatness should be discouraged; first of all the boy tends to abbreviate his finished copy and so to leave out essential parts of the work, secondly he may make mistakes in copying, and thirdly it is a waste of his time. Boys who do this defend themselves on the ground that they do not want to spoil the neatness of their work by showing up unsuccessful attempts, but I tell them that in my eyes an unsuccessful attempt that is neatly crossed off does not detract from the neatness of their work.

THE USE OF SCRAP PAPER

Whenever I am teaching, whether or no I am working on the board, I like each boy to have beside him a sheet of scrap paper; on that he can write down the answer to any question which I ask the class generally, or he can attempt the next step of the work which I am doing on the board, or draw a figure for himself if I am going to do a piece of geometry.

This serves two purposes: first, if a boy is liable to be called on at any moment to write down an answer or do the next step of a piece of work being done on the board, he must pay active attention and need not merely look interested; secondly, if he makes a mistake he knows that he has done so, or if he is uncertain of a step he is aware of it, and consequently he is much

more likely to ask a question about it and get it cleared up at once.

So strongly do I feel about the proper use of scrap paper that I shall return to the point under Algebra (see p. 187) and Geometry (see p. 258).

SHOULD WORK BE ALLOWED IN THE MARGIN?

I was asked this question some years ago by a preparatory school headmaster. It appeared later that he was not a mathematician, the mathematicians on his staff were divided on the point, and he wanted a statement that would enable him to give a ruling.

My answer ran somewhat as follows:

In general it is nice that the answer to a question should be complete without any side work; but, where a piece of computation breaks the thread of the argument, it is far better that the computation should be put at the side, for example, a square root that arises in solving a quadratic equation should be put at the side. But

(i) never allow work on another sheet of paper;

(ii) work at the side must be neat, legible and intelligible;

(iii) it should be done opposite the place at which it is wanted so that it is easy to refer to;

(iv) on no account should it be called "rough work," or it will be done untidily and so inaccurately.

He went on to ask whether work in the margin is allowed in more advanced mathematics.

Of course it is used a great deal. In the course of a piece of mensuration perhaps logarithms have to be used or long multiplication to find an area; the mere calculation is not germane to the argument and may well go to the side of the paper, but it must be shown and must be clear and intelligible and must not be done on scrap paper.

MISCELLANEOUS

When tackling a new topic, the teacher should aim at showing the use of it. If possible, it is nice to let the topic arise out of some practical problem that has arisen. Often it is wise just to break new ground and then show something of the vista that it opens up; much of what the master says will not be completely followed by any of the class, but it will interest them and show what they may reach by the work that is immediately before them.

Again I am quite certain that, when a boy is given a new idea, a process goes on inside him which I like to call "subconscious digestion." Let me give an example from my personal experience. When I was a comparatively young teacher, it fell to my lot term after term to start many boys on logarithms and fractional indices. On one occasion, I found that I had a few minutes to spare at the end of a lesson, and I said, "Next time we are going to start on fractional indices; we have a few minutes left today, so we will start now."

I went through the usual process of considering

$$10^p \times 10^q = 10^{p+q}$$

for integers and then assuming it true when p and q are fractions, and I got as far as showing $10^{\frac{1}{2}} = \sqrt{10}$; then it was time to go. The next time that I could take the subject up again happened to be two or three days later; and then it struck me that the lesson went very much better than my usual first lesson on the subject. Naturally I tried to think why the lesson had gone better than usual; I came to the conclusion that there had been some "subconscious digestion" of the idea which I had given two or three days before. Or to give a different simile, I had sown a seed in the boys' brains and its case had been a little softened in the few intervening days; or yet another simile, I wanted to plough a certain field and a few days before I had scratched the surface enough to let in some

moisture so that the ploughing was easier. Ever since that experience I have always felt that there is a time factor for the boy for digesting new ideas, and that it pays just to start a boy on a new topic and then leave it for a few days before coming to real grips with it.

Another point, what is the right course to pursue when the class seems to have got rather stuck on some subject—their interest is perhaps beginning to flag and they make the same mistake again and again? The really stout-hearted teacher may be tempted to say "we will have such a grind at this that every boy shall be perfect at it and we won't do anything new till they are"; but is that the right course? Sometimes it is, but more often the better course is to leave the subject alone for a week or two, and work at something entirely different; then go back to the difficulty, start some little way back, go through the whole thing carefully, take easy examples and the class will probably go right past the old sticking point with ease. The explanation is probably only that they were getting a little stale, though sometimes it is that there was some earlier point that they had failed to grasp, though the teacher had not realised it, and that he cleared it up on the second course.

In the last few pages I have mentioned some of the things that I have learnt from my classes. No teacher is too old to learn from his classes; he should always be observing and getting new ideas for the presentation of the different parts of his subject.

Finally I would urge every teacher to be an enthusiast and be cheerful. Years ago I knew of a teacher who used to say "Now we have got to do these horrid dull things," and they proved horrid and dull; if the work was essentially dull, my attitude would have been "Now we have got to do so and so, it may not be very attractive but it is necessary and useful, let us concentrate hard on it and get it really good, so that we can go on to something that is more attractive."

One more point, a schoolmaster has to devote so much of his time to finding mistakes that he is apt to forget that praise and encouragement are what many boys really want. Always try to find opportunities for praising any good work or even any improvement in a boy's work.

BLACKBOARDS

A necessary part of the equipment in every mathematical class-room is a large blackboard: it should be at least 6 feet long. Part of it should be ruled in 2-inch squares (smaller squares are too small). The best way of doing this with most blackboards is to get the carpenter to make cuts along the necessary lines with a sharp knife; in a week or two enough chalk will have worked into the lines to make them show up.

Many masters' writing on the board is too small. Boys' attention will flag more quickly if eye strain is needed to read the writing.

"THE NINE QUESTIONS"

See p. 59.

Papers 1–4 were set to a class of average age 10 yr. 1 m.

Paper 1

2*s*. $3\frac{1}{2}$*d*. × 12	*s*. *d*. 5 $4\frac{3}{4}$ 7	*s*. *d*. 7 | 17 $3\frac{1}{2}$
15*s*. 3*d*. ÷ 12	60 × 50	9*s*. 3*d*. − 3*s*. 9*d*.
113 pence in *s*. *d*.	11*s*. 3*d*. in pence	Add up all the numbers up to 10

Paper 2

$2 \times 3 \times 5 \times 7 \times 11$	54 in prime factors	Prime numbers between 30 and 40
$3^2 - 2^3$	$3 \times 5 \times 7 \times 11 \times 0$	50×40
5 watches cost £45 15 watches cost	5 men take 45 days 15 men take	Add up the ten numbers from 31 to 40

Paper 3

Prime numbers between 50 and 60	91 in prime factors	$2 \times 11 \times 13$
$2^5 - 5^2$	20^2	$\sqrt{121}$
2*s*. 5¼*d*. × 7	12*s*. 5¼*d*. ÷ 7	Orchestra of 20 take 6 min. to play it. Orchestra of 40 would take

Paper 4

A ludo board has 12 squares along each side. How many squares altogether?	A halma board has 121 squares altogether. How many along each side?	$5 \times 7 \times 13$
19*s*. 11*d*. in pence	200 pence in *s*. *d*.	53 in prime factors
How far to walk round a hockey field 100 yd. long and 50 yd. wide?	12 men take 144 days 3 men take	Prime numbers between 20 and 30

Papers 5–9 were set to a class of average age 11 yr. 10 m.

Paper 5

$2 - \cdot 2$	$\cdot 5 \times \cdot 6$	$\cdot 05$ as a vulgar fraction
$2\cdot 37 \times 10$	$\cdot 21 - 7$	$1\frac{1}{5}$ as a decimal
$2\cdot 37 \div 100$	$\frac{2}{3}$ of £1	$(1\frac{1}{2})^2$

Paper 6

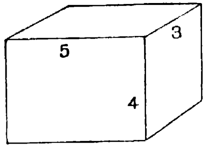
$-3x = -21$ $\therefore x =$	$-35 \div (-7)$	$10x - (3x - 4)$
$-3\frac{1}{2}x + 2\frac{1}{4}x$	$x^2 + x + 5$ when $x = \frac{1}{2}$	$1\frac{1}{5}\sqrt{H}$ when $H = 25$
We scored x goals; they scored x tries. We won by	$2x + 2 = 20 - 4x$ $\therefore x =$	$a^2 - 2a$ when $a = 3$

Paper 7

$\frac{c^2}{8}$ $c = 10$	ab $a = \frac{2}{3}, \quad b = \frac{3}{4}$	$3x - 3 + 5x$
$16x - x + 10$	$2x - \frac{1}{2} = \frac{3}{2}$	x goals y tries = points
How many pounds in x shillings?	$8 - 7 + 6 - 3$	$2x^2 + 3x + 2$ when $x = 5$

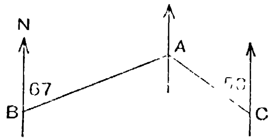
Paper 8

Wire



Calico

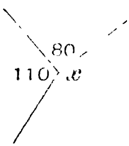
Air

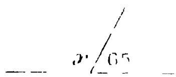


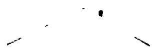
Bearing of A from B

Bearing of C from A

Angle S. 57° E. makes with the North

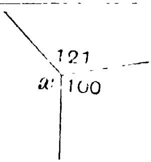


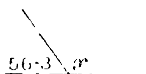


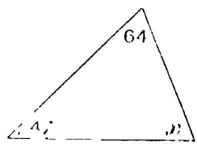


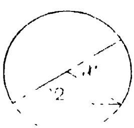
Kind?

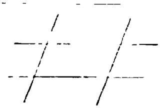
Paper 9





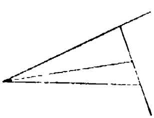




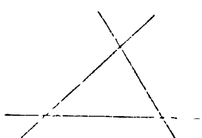


How many pairs of alternate angles?

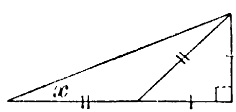
How many pairs of corresponding angles?



How many pairs of supplementary angles?

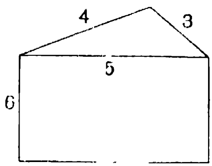


How many pairs of vertically opposite angles?



Papers 10–15 were set to a class of average age 12 yr. 9 m.

Paper 10

$3\frac{1}{2}$ % as a fraction	$\frac{7}{8}$ as a percentage	$(5\frac{1}{2})^2$
 Volume	Surface	Divide £10 in the ratio 4:5
Man takes 3 hours Boy takes 4 hours Man and 2 boys take	Goes at 6 m.p.h. Returns at 2 m.p.h. Average speed	·234 × ·007

Paper 11

5 % of £23. 7*s*. 6*d*.	Simp. Int. on £25. 10*s*. for 3 years at 5 %	Comp. Int. on £300 for 2 years at 10 %
65 % as a v. fract.	$\frac{5}{8}$ as a percentage	£3. 17*s*. 6*d*. as a decimal of £1
£3·625 in £ *s*. *d*.	4 Redheads in the class. How many %?	$\sqrt{15^2 - 9^2}$

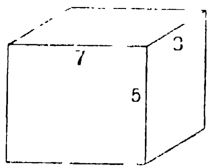
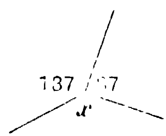
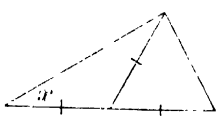
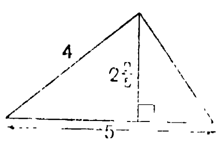
Paper 12

$(x-2)(x+3)=0$ $\therefore x=$	$(2x+5)(3x-2)=0$ $\therefore x=$	$3(x-1)(x+3)=0$ $\therefore x=$
$x(x+9)=0$ $\therefore x=$	$x^2-3x=0$ $\therefore x=$	$x^2-5x+6=0$ $\therefore x=$
$x^2-5x=6$ $\therefore x=$	Expand $(x-7)^2$	Expand $(x+\frac{1}{2})^2$

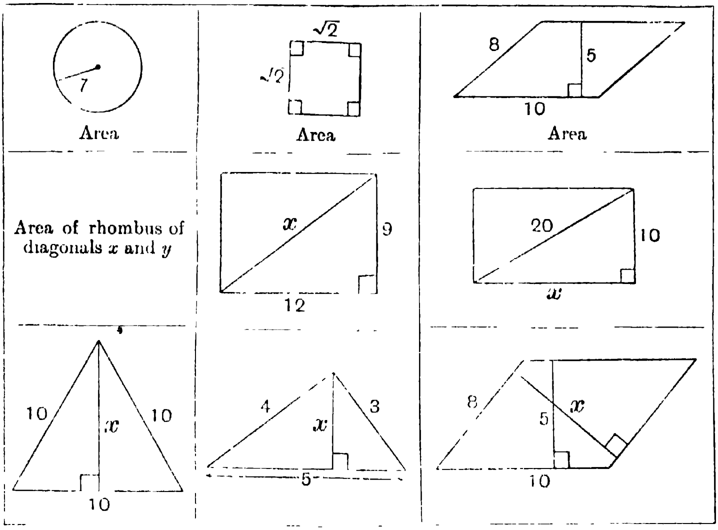
Paper 13

$x = 2 \pm 1{\cdot}24$ $\therefore x =$	$x = -4 \pm 2{\cdot}36$ $\therefore x =$	Complete $x^2 - 3x \qquad (x \qquad)^2$
Equation with roots 2 and $-\frac{3}{4}$	Roots of $5x^2(3x+4) = 0$	Roots of $x^2 + 5x - 3 = 0$ are $\cdot 54$ and $-5{\cdot}54$ Factors of $x^2 + 5x - 3$ are
Factors of $4x^2 - 64y^2$	Divide $y^2 - 2y$ by $y - 2$	Expand $(2x - \frac{1}{2}y)^2$

Paper 14

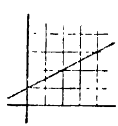
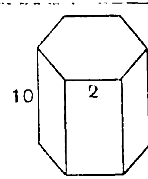
 Calico	A from B is N. 23° W. B from A is	
	Int. $\angle$ of regular octagon	How many diagonals has it?
 Area	Diagonals equal and bisect each other at right angles	Angle of 327° Is this possible for a regular n-gon?

Paper 15



Papers 16–20 were set to a top class of average age 13 yr. 6 m.

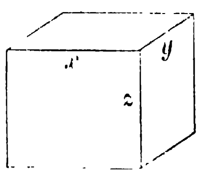
Paper 16

$\sqrt{50}$	$2=10^{\cdot 3010}$ $\therefore\ 20=10^x$	$32=10^x$
The price is 10*s*. after being lowered 10 % Find the original price	 Equation	 Volume
$2^{10}=1024$ $\therefore\ (\cdot 2)^{10}=$	*A* takes 1 min. *B* takes 1 hour $\therefore$ Together	Divide 4*d*. in the ratio 7:9

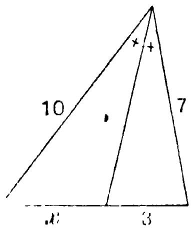
Paper 17

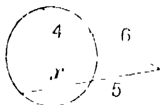
Amount at C.I. of £300 for 2 years at 10 %	Amount at S.I. of £300 for 7 years at 5 %	Expression for finding amount at C.I. of £235 for 20 years at 4 %
$\sqrt{3} = 1.732$ $\sqrt{48} =$	$\frac{1}{\sqrt{3}} =$	Expression for finding amount at S.I. of £235 for 20 years at 4 %
log 2 = ·3010 log 12·8	1 ÷ ·50 =	Curved surface of smallest cylinder that will contain a sphere of radius 7″

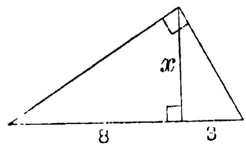
Paper 18

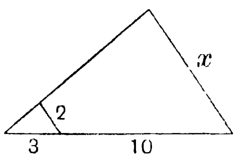
$9^{\frac{1}{2}}$	$\left(\frac{16a^2}{9b^2}\right)^{-\frac{1}{2}}$	$2 = 10^{\cdot 3010}$ $\sqrt{2} = 10^{x}$
$S = 2\pi r(r+h)$ $h =$	Sum of roots of $5x^2 - 3x - 7 = 0$	$3 = 10^{\cdot 4771}$ $\frac{2}{3} =$
$\frac{5}{3} \quad \frac{2}{x}$	Factorise $a^2 + 2ab + b^2 + a + b$	 Diagonal

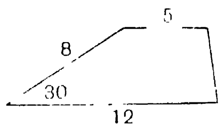
Paper 19











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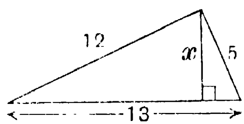
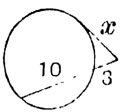
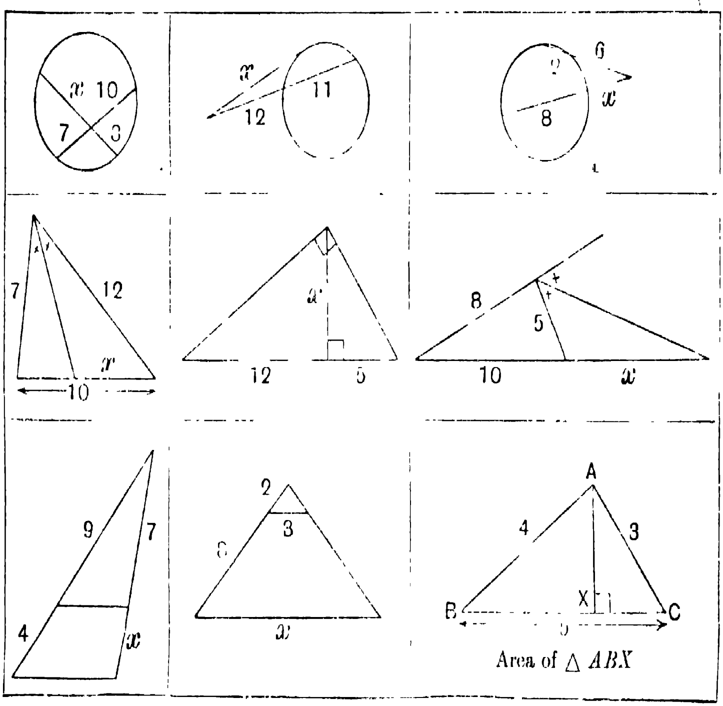




Angle of regular dodecagon

ow many diagonals'

Paper 20



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PART III

ARITHMETIC

BY A. W. SIDDONS

The references are to Godfrey and Price's *Arithmetic* (G. and P. *Arithmetic*). See also a list of books by Godfrey and Siddons, etc., on page 323. See also Siddons, Snell and Lockwood's *A New Arithmetic*

CHAPTER I

PRELIMINARY

When we are considering the methods we should adopt in teaching a subject, it is well to keep before our minds our reasons for teaching it. In the case of arithmetic there are two reasons, (i) the subject is valuable in itself, in fact it is essential in our modern world, (ii) it is a means of drawing out and developing a child's mathematical powers.

In teaching the fundamental processes in arithmetic we must not forget that we are using them as a means of developing the child's natural ability, and we must teach accordingly. On the other hand, these processes will be used so much throughout the child's life that it is desirable that our teaching should lead straight on to the methods which give the greatest quickness and accuracy; further than that, these processes must ultimately become mechanical, and the teaching of a variety of methods for one process will cause delay in that process becoming mechanical, so that it is wise that the child should be taught as early as possible to use the methods which he will finally use. But that is no excuse for making them mechanical too soon, or for not appealing at every stage to the child's reason; but all the time there must be an accompaniment that does not appeal to the reason at all, namely the mechanical dealing with numbers—the calculating machine activity.

When once the mechanical parts of arithmetic are mastered, there should be no more rules. It is really astounding to find teachers still giving rules for the solutions of particular types of problems which are quite unimportant in the outside world; by so doing they destroy the child's initiative in attacking any question that is new to him, and they cumber his memory with useless stuff which later gives him a contempt for the subject as

taught in schools. A great teacher used to say "There are five rules in arithmetic—addition, subtraction, multiplication, division and COMMON SENSE."

It must not be inferred from the above that arithmetic can be divided into two parts, viz. (i) the fundamental processes, and (ii) their application, and that the first part can be finished off before the applications are begun. At every stage there must be applications, and problems; in fact, the fundamental processes themselves should gradually grow out of problems.

It is not my intention to deal fully in the following chapters with the "Groundwork of Arithmetic"*; nevertheless, I shall refer to some things that come into that stage, for the sake both of the teachers who are doing that work, and also for the teachers who are doing the later work. It is very important that these two classes of teachers should keep in touch with one another; otherwise, it is impossible for either to make the best of his opportunities. For instance, there is much excellent teaching being done in kindergartens and by nursery governesses, and it will be of real help to the masters in an elementary school or in a preparatory school to know of that teaching. Again, the teacher of the elementary work will be able to prepare the children better for the later work if she knows something of the work done and methods used in the later work.

I have referred above to the excellence of much of the kindergarten work. But, sad to say, much of the work at that stage is still very bad and old-fashioned; and this weakness would be corrected to some extent if teachers would keep in touch with the methods used later on.

One golden principle throughout the teaching of arithmetic should be—"Do not give unnecessary rules." To this might be added two corollaries: "Do not teach anything that has to be unlearnt later"; "Do not give a rule for a special case when that will be included in a general case to be considered later."

* There is an excellent book with that title by Miss Punnett (Longman's).

Some of the most striking violations of the above principles occur in the teaching of algebra; but here are a couple of rules, frequently taught in arithmetic, which give trouble when the boy gets on to the same type of work in decimals—

"To divide by 10, cut off the last figure," and, "In finding the square root of a number, first mark off the figures in pairs starting from the right." There can be no excuse for the latter rule, as a boy has generally done decimals before he does square root, so that he might just as well learn from the first to mark off in pairs from the decimal point.

Good teaching of arithmetic is not only important for the sake of arithmetic, but also because it paves the way for algebra. In the chapter on "Vulgar Fractions" it is pointed out that, if this work is well done, fractions in algebra are a simple matter. Again, letters should be introduced into arithmetic (see "Algebra," chap. IV, p. 169); this use of letters in arithmetic, apart from its help in algebra later, helps the arithmetic—if a boy after expressing, 1*s*., 2*s*., 5*s*., in pence is asked to express *xs*. in pence, the generality of the process is brought home to him; such examples may not appear in the arithmetic book, but the teacher should easily supply them out of his head.

DRILL IN ARITHMETIC

We have got to recognise that a large part of arithmetic has to become mechanical and is not of much use practically until it has become so; consequently it is essential that there should be plenty of drill, but the drill must be kept bright and must be taken in small regular doses. If the class expect this at the beginning of each lesson and regard it as a necessary duty to be done quickly and cheerfully before they can get on to the more interesting work, they will not regard it as dull. The time given to this drill should be limited: it must not be allowed to take too much time from the lesson. On some days five minutes will be enough, occasionally it may be good to give as much as

ten minutes to it. Sometimes it is desirable that they should have no feeling of rush and the number of sums given should be such that all can finish them in the time allotted; in such cases it is necessary for those who have finished to be employed and they should be taught to use their time, they should turn up their arithmetic or geometry books and read these. At other times it is desirable to put up enough sums to keep all employed the whole time; then it is necessary to choose the early sums carefully so that the slower boys do some of all the types it is desired to revise, the later sums should be heavier examples of the same type as the earlier ones.

Whether the master reads off the answers at once, has the sums marked right or wrong and deals there and then with the mistakes will depend on circumstances; sometimes that is the best course, at other times he should mark them out of school.

Drill work needs careful thought and preparation; the master must think out what types of sums he wants to revise; perhaps he will make a list and put a tick against each item when he has had enough drill at it.

CHAPTER II

SPEED AND ACCURACY IN ARITHMETICAL COMPUTATION

It should be unnecessary to plead for the importance of speed and accuracy in computation; but it is appalling to find how many boys of 14 are slow and inaccurate in simple addition and in multiplication tables; this is sometimes attributed to a lack of natural ability for figures, but I believe that in general it is due to bad teaching and can be cured, though it is easier to cure at the age of 10 than at 14 or later. I wonder if it is realised how much time would be saved between the ages of 10 and 18, say, if a child could halve the actual time he took over mere computation.

The necessary drill can be made interesting, and it does not take a great deal of time (5 minutes a day is generally enough) but it should be regular, and it needs to be kept bright so that the child is really attracted and concentrates on it willingly.

Three questions naturally arise, (i) At what age should the necessary training be given? (ii) What should it be? (iii) How should it be given?

The question of the proper age is one for the psychologist; but I am quite clear that, if a child of 9 or 10 is slow and inaccurate, the training should not be postponed.

Now what is the best drill for the purpose, and how should it be given? I am strongly of opinion that it should be almost entirely *vivâ voce*; to give it with written work produces dullness, deprives the teacher of his best opportunity of discovering the child's methods, and takes more time without producing as good an effect. Of course there must be some written work too, but the main drill should be *vivâ voce*. I hear the reader who has never tried it muttering to himself that there would be more

waste of time with the *vivâ voce* work because only one child works at a time, and that you cannot mark *vivâ voce* work so well; to the former objection I would reply that it is not true, at any rate if the teacher is at all skilful; and to the latter, there is no need to mark everything a child does, and useful variety is obtained by occasionally making each child write down his answers—this work can be marked.

Before I come to details, I should like to mention two general points. First of all a child of 10 must not be allowed to count on his fingers. Secondly, *vivâ voce* work, if every child is kept going, is tiring, and should only be done for about five minutes at a time; in an arithmetic lesson of forty minutes I would suggest five minutes at the beginning and, perhaps, five minutes at the end sometimes. This work should be done every day, until reasonable speed and accuracy are acquired; it should become the regular habit of the class to expect it at the beginning of every arithmetic lesson, and they should have paper or books ready to write down some of the answers when asked to do so.

Of course there should be mixed problems and the drill in the mechanical part should be varied. The following suggestions are arranged in subjects, but it is not expected that a teacher would take them in order, or even take all of them; they are merely suggestions from which the teacher can select.

Each child should be provided with a printed or "jelly-graphed" copy of a square of figures such as one of the following:

(i)

6	4	2	9	3
3	5	8	7	6
2	0	6	3	8
4	9	0	8	4
8	3	5	1	9

8	5	0	3	9	7	6
5	1	9	1	8	3	4
2	6	4	8	0	7	3
0	8	3	2	7	3	0
6	4	6	5	8	7	4
7	5	8	1	2	0	5
4	7	0	6	4	8	9

(iii)	6	5	9	6	7	8	2	7	1	3	8	9
	3	5	1	9	4	4	4	9	3	0	6	9
	3	3	2	4	0	6	9	2	4	0	0	1
	8	1	3	1	8	0	2	8	5	9	5	4
	5	7	4	6	0	9	0	8	5	6	3	3
	1	0	4	7	3	0	1	9	6	8	6	0
	0	2	7	6	2	5	9	7	5	2	8	1
	7	2	2	4	9	0	3	6	7	7	0	5
	3	6	5	8	5	6	5	9	0	8	2	2
	6	2	9	3	4	6	8	6	4	8	5	0
	1	7	8	7	3	5	1	4	7	2	1	8
	8	5	7	1	9	2	1	2	7	1	4	5

The figures should be well spaced and clear; when once printed or "jellygraphed" they will last for years; suitable sets ought to be printed in every arithmetic book.

ADDITION AND SUBTRACTION

A. (i) Count straight forward by 1's thus: 1, 2, 3, 4, ...

(ii) Count forward by 2's, thus: 1, 3, 5, 7, ...; or 2, 4, 6, 8, ...

(iii) Count forward by 3's, thus: 1, 4, 7, 10, ...; or 2, 5, 8, 11, ...; or 3, 6, 9, 12, ...

(iv) Also by 4's, 5's, ..., 9's.

B. Count backwards in the same way, thus:

(iii) 50, 47, 44, 41, ...

C. Write on the board in large well-spaced figures:

2, 7, 3, 5, 1, 6, 4, 9, 8.

State the result of adding 4 to each of these numbers; repeat adding (instead of 4) other numbers up to 9.

D. Repeat *C*, adding (instead of the 4) any numbers less than 100, thus for 37 the results would be:

39, 44, 40, 42, 38, 43, 41, 46, 45.

E. Cover all but, say, the first two columns of the square (iii) (we now have a lot of numbers of two figures each), add 8 (or any other number less than 10) on to each of the numbers we have.

F. Using the figures in *C*, state the result of subtracting 4 from each number; in the case of 2 (or any number less than 4) the child must imagine a 1 supplied in front of it; thus the answers with the above figures would be:

8, 3, 9, 1, 7, 2, 0, 5, 4.

G. State the result of taking each of these numbers from 33, say, thus:

31, 26, 30, 28, 32, 27, 29, 24, 25.

H. Add up continuously from left to right, or right to left, or as the master points, saying aloud the result of each addition; thus, adding from left to right, the child would say:

2, 9, 12, 17, 18, 24, 28, 37, 45.

I. Subtract in the same way, starting at 100 or any smaller number; thus, from left to right:

98, 91, 88, 83, 82, 76, 72, 63, 55.

J. Using the numbers in *C*, and working from left to right, state what must be added to each number to make the next (an extra ten being supplied when necessary); thus:

5, 6, 2, 6, 5, 8, 5, 9.

K. Repeat each of the above, using the rows or columns of the squares.

L. Repeat, assuming that the numbers represent pence, and add or subtract in shillings and pence.

In all this work a child must get out of the way of saying "Five and seven make twelve"; at the sight of $5 + 7$ he must learn to say 12 immediately. In the same way, $8 - 3$ must suggest 5 at once, there should be no need to say "3 from 8 leaves 5," or any such form of words.

MULTIPLICATION

Tables should be so well known that there is no need to say "9 times 6 is 54"; 9 × 6 should suggest 54 at once, and 9 times 6 and 6 times 9 should be equally familiar. Here are a few suggestions for *virâ voce* work:

M. Take a set of numbers as in *C*, and state the result of multiplying each number by 6.

N. Take a set of numbers as in *C*, and state the product of each consecutive pair, thus:

14, 21, 15, 5, 6, 24, 36, 72.

(An occasional 0 is useful here.)

O. Or, again, state the product of any two numbers that are pointed to.

P. With the figures in *C* multiply each figure by 4, and add to the product the next figure to the right; thus:

15, 31, 17, 21, 10, 28, 25, 44.

At first it is best to name the product as well as the result of adding on the next figure, thus:

8, 15; 28, 31; 12, 17; etc.

Q. With the figures in *C* state the product of each pair with the next figure added to the product, thus:

17, 26, 16, 11, 10, 33, 44.

R. Repeat each of the above with the lines and columns of the squares.

DIVISION

There is probably no better practice than ordinary short division; the rows in the squares can be used for this purpose.

I have only put out the above as suggestions; the list is not exhaustive, and is not meant to be followed rigorously. All the

processes mentioned will be found useful, I think, in the subsequent written work; but, though they should prove helpful in saving some of the time devoted to written work, they cannot be substituted for it entirely. Both the *vivâ voce* work and the written work help to show up which children need most drill.

Sometimes it is wise to have drill on a particular number, but in general it is variety that is wanted.

A useful variation is to draw a clock face; (i) add, say, 4 on to each number as the teacher points to them, (ii) add up continuously as the teacher points, (iii) state the result of multiplying each number by 5, etc., etc.

Another variation is to give an addition sum thus: the teacher reads off the numbers and the class add them as they are read, at the end each child writes down his answer; this can be varied by making the lesson a shopping expedition and counting up how much money has been spent. This can be made a somewhat severe test of concentration.

I remarked above that in *vivâ voce* work it should not be the case that only one child worked at a time; a skilful teacher should be able to keep every member of a class at work, provided he does not work them too long at one time (a fatal mistake). Attention can be kept, if the children are fresh, by dodging the questions about the class, by letting one child go on with almost any of the above exercises and then suddenly turning on another child, and by keeping them on the look-out for each other's mistakes; if the children are not fresh, it is impossible to keep them all going, and probably it is only nature's safeguard that makes them inattentive in that case. It is interesting occasionally to time different children, say, at adding a column of the squares above; their times at the beginning and end of a term are often instructive; if the number of mistakes is also recorded, the whole class are sure to be very attentive—there is a cruel joy in spotting another's mistake.

It must not be supposed that quickness and accuracy will be

acquired in a single term. Steady training has a wonderful effect, but it must be kept up for some time or the results will be transient.

Any teacher who has not done much work of this type will probably find his first term of it very interesting, and it is unlikely that his interest will flag. As to the interest the class will show, all depends on the teacher; a teacher with zeal and personality can make a class enthusiastic about this work, but he will have to vary his methods and sometimes give marks or some indication of progress.

CHAPTER III

FOUR RULES, SIMPLE AND COMPOUND

In this chapter I am not going to deal with the first teaching of the four rules, but I should like the nursery governess or kindergarten teacher to read the chapter so that she may "teach for the future" and realise the attitude of the teacher who takes her pupils after, say, 9.

My somewhat random notes will refer to the ultimate methods to be used. The sooner a child gets on to these and can use them mechanically the better, *provided he understands the rhyme and reason*. The early work must of course be concrete; the abstract drill described here should come after the desire for mechanical skill has been developed.

ADDITION

All addition ultimately depends on the power to add one number less than ten to another number less than one hundred. This power must be acquired by *vivâ voce* work, and must be ground at until perfect. See chap. II, "Speed and Accuracy."

Children should learn to check their work. If they add from the bottom upwards, the check should be made by adding from the top downwards (otherwise a mistake is liable to be repeated). Each child should get into the habit of always going the same way for the first addition; it is quite immaterial which way that should be; probably most people make the first addition upwards.

In adding mentally two numbers such as 37 and 25, it is simpler to take one of them, say 37, and add on first the tens of the other number, and then the units. Thus

$$37 + 20 = 57, \quad 57 + 5 = 62.$$

SUBTRACTION

There are a surprising number of different methods of subtraction. I am going to refer to two only, as I think all children should use one of these two.

The Method of Complementary Addition

Before teaching this there must have been plenty of *vivâ voce* drill of the type

$$5 + ? = 8^*, \quad 9 + ? = 16.$$

I do not propose to deal further with that stage.

In case any of my readers are not familiar with the method of complementary addition, I will use it for a couple of subtraction sums, showing the argument in full.

Start with an addition sum thus

```
5 7 8
3 8 6
-----
9 6 4
```

Rub out the second line thus

```
5 7 8

9 6 4
```

and try to find what has been rubbed out.

The argument runs as follows—the bold figures being the ones to write down. 8 and **6** make 14, 8 (i.e. 7 and 1) and **8** make 16, 6 (i.e. 5 and 1) and **3** make 9.

Now take an example set down in the usual form

```
From  1 3 8 4 4 8
Take    5 4 6 8 3
```

The argument is 3 and **5** make 8, 8 and **6** make 14, 7 and **7** make 14, 5 and **3** make 8, 5 and **8** make 13.

* Read "5 and what make 8".

The Method of "Taking Away"

There are at least two methods in use here.

The argument runs as follows:

Units: 8 from 4, you cannot; 8 from 14, 6			9 4
			5 8
Tens:	*1st method*	*2nd method*	
	6 from 9, 3	5 from 8, 3	

The first method is sometimes called "the method of equal additions" (a ten is added to the top line to make the 14, and a ten added to the bottom line changes the 5 tens to 6 tens). A small child once suggested the following to me, and it seems to me simpler than the usual explanation:

To make the 4 up to the 14 we have to take 1 ten from the 9 tens, we have also to take 5 tens from the 9 tens, so we take 5 + 1 tens, i.e. 6 tens from the 9 tens.

The child's words were: "You take the 1 ten away at the same time as the 5 tens."

The second method for dealing with the tens in this case is sometimes called "the method of decomposition"—the 9 tens are broken up into 8 tens and 10 units—so that first 1 ten is taken away from the 9 tens leaving 8 tens and after that the 5 tens are taken from the 8 tens.

It is often argued that the method of equal additions is harder to explain and I agree that most of the explanations given are hard for the child, but the one I have given above seems to be perfectly simple, and children understand it at

* In the kindergarten the method of decomposition is generally used, at any rate at first; and I think this is right, but after very few lessons I would advocate changing to the method of equal additions with the explanation I have suggested above.

Besides considering which method appeals to the child's reason, we must also consider which ultimately leads to the greater accuracy. Both methods have been submitted to most careful tests, and it has been shown that the first method (the so-called equal addition method) leads to greater accuracy and greater speed.

In examples such as 1000 – 357, or £5. 0*s*. 0*d*. – £3. 4*s*. 8*d*. (types which occur in actual practice much more frequently than in text-books), one's debts (if one uses the term "borrow") are settled at once in the equal addition method, even if one runs into debt again; in the decomposition method the borrowing is continued at any rate till the 0's are done with.

SIMPLE MULTIPLICATION

All multiplication ultimately depends on being able to state accurately the product of any two numbers each not greater than twelve.

Here again tables must be perfect and must have much *vivâ voce* practice. See chap. II, "Speed and Accuracy."

In long multiplication always multiply first by the left-hand figure of the multiplier. This is most important. I was much surprised some years ago to find a large number of future teachers had not been taught to do this, and again a large number had been taught to do it for decimals, but did not do it when multiplying integers.

The question may arise whether a child should be made to change if he has learnt to multiply by the right-hand figure first. I should say it all depends on the age of the child: I should certainly advise that a child of 9 or 10 should be made to multiply by the left-hand figure first, but I should hesitate about compelling a child of 12 or 13 to change his method.

Care should be taken that the right-hand figure in each product falls below the figure used as a multiplier.

```
Thus        9573        or     9573
            3058                 3058
       ---------            ----------
       28719                28719
          47865                47865
            76584                76584
       --------             ---------
       29274234             29274234
```

COMPOUND MULTIPLICATION

Let me quote from the *Mathematical Association Report on Teaching in Preparatory Schools*, 1924.

"Some method for long multiplication of £ *s*. *d*. should be taught in addition to the practice method. Either the 'ten ten' method, or the method in which the numbers of £ *s*. *d*. are used as multipliers.

The method of 'practice,' in which division is substituted for multiplication, should be regarded as a shorter alternative method."

The following arrangement for the method, in which the numbers of £ *s*. *d*. are used as multipliers, I believe to be new*. Compare Prof. Nunn's arrangement for division, see p. 100.

To multiply £15. 14*s*. 9*d*. *by* 365.

```
    £             s.                d.
    15            14                 9
   365           365               365
  -----------------------------------------
  365           365           12)3285
  1825          1460  <-        ----273s. 9d.
   269<-         273<-
  -----
  £5744  |  20)5383
         |  ---- £269. 3s.

                                £5744. 3s. 9d.
```

* This chapter was written in 1924. I evolved the method as the result of setting the following question in an examination for Teacher's Certificates:

Note that in the case of the £'s, the 365 is multiplied by the 15, instead of the 15 by 365, and similarly with the shillings and pence.

When the method is thoroughly familiar, it might be arranged as follows:

```
      £            s.          d.
      15           14           9
     365          365         365
   -----        -----       -----
     365          365    12)3285
    1825         1460   |  ------ 9d.
     269 <-       273 <-'
   -----   |
   £5744   | 20)5383
   -----   |    ----
           |       3s.
```

The practice method is so familiar that I need not give an example; but I must point out how frequently it is used, e.g. to calculate the income tax at 4*s*. 6*d*. in the pound on £563. 7*s*. 4*d*., the easiest method is to take $\frac{1}{5}$ of the sum (4*s*. = $\frac{1}{5}$ of £1) and $\frac{1}{8}$ of that result (6*d*. = $\frac{1}{8}$ of 4*s*.) and add the two results.

DIVISION

Many children have a pernicious habit of saying "3 into 12" when dividing 12 by 3. This leads to confusion in algebra, e.g. $5\,(a + b)$ is read "5 into $(a + b)$" and "into" here is equivalent to "multiplied by" or "times."

$12 \div 3$ can be read "Divide 12 into 3 parts" or "12 divided by 3."

In long division the quotient should always be put over the top of the dividend, the first figure of the quotient being written immediately above the right-hand figure of the first product.

"Find, by two distinct methods, the value of 365 times £2. 11*s*. 9*d*." Dr Ballard quite independently arrived at an almost identical method and published it in his work *Teaching the Fundamentals of Arithmetic* in 1928.

To divide 49769 *by* 253.

```
          196
    253)49769
        253
        2446
        2277
         1699
         1518
          181
```

Quotient 196, remainder 181.

COMPOUND DIVISION

The following example shows the neatest arrangement I know for the process—it is due, I believe, to Prof. Nunn.

The work is compact and orderly, all the pounds are in one column, all the shillings in another, etc., and the arrangement appeals to quite young children.

To divide £1290. 2*s*. 3*d*. *by* 73.

```
        £                    d.
        17         13                  and 34 d. over
  73)1290                                    ↑
        73      →980       →396
        560      982        399
        511      73         365
         49—     252         34
                 219
                  33—
```

CHAPTER IV

THE UNITARY METHOD AND ITS DEVELOPMENTS

The majority of present-day teachers must have been brought up on the Unitary Method, though there may be a few who learnt "Rule of Three" in the days of their youth; but I suppose all teachers would agree now that "Rule of Three" should be a thing of the past.

FIRST METHOD

I will give a couple of examples of plain honest Unitary Method

Ex. 1. *If an aeroplane travels 648 miles in 9 hours, how far will it travel in 6 hours at the same rate?* *

* [The first thing is to ask what is the ultimate question; it is "How many miles?" Then consider the wording of the last line and arrange it so that the number of miles comes at the end. It will be "In 6 hours the aeroplane travels — miles." Now word the given statement accordingly.]

In 9 hours the aeroplane travels 648 miles,

$\therefore$ in 1 hour „ „ $\frac{648}{9}$ miles,

$\therefore$ in 6 hours „ „ $\frac{648 \times 6}{9}$ miles†,

$$= \frac{\overset{216}{\cancel{648}} \times \overset{2}{\cancel{6}}}{\underset{3}{\cancel{9}}} \text{ miles,}$$

$= 432$ miles.

* The part in brackets not to be written down.

† I would urge teachers to insist on this being repeated before any cancelling is done, on the grounds that this wants to be read in looking over the work and it often becomes illegible with cancelling. Teachers should insist on cancelling work being kept neat.

Ex. 2. *If a ship is coaled by 200 men in 10 hours, how long will it take 250 men to coal the ship?*

[The ultimate question is "How many hours?" The last line will read "250 men coal the ship in — hours."]

200 men coal the ship in 10 hours,

$\therefore$ 1 man coals ,, 10×200 hours*,

$\therefore$ 250 men coal ,, $\frac{10 \times 200}{250}$ hours,

$$= \frac{\overset{40}{\cancel{10} \times \cancel{200}}}{\cancel{250}} \text{ hours} = 8 \text{ hours.}$$

SECOND METHOD

The first method should be used generally for 2 or 3 years; but later on it may be shortened thus†:

Ex. 1. In 9 hours the aeroplane travels 648 miles,

in 6 ,, ,, $648 \times$ — miles‡.

Teacher. Now from previous experience we know that the 648 hours has to be multiplied by one of the numbers 6 and 9 and divided by the other. Has the number of miles to be made larger or smaller?

Pupil. Smaller.

Teacher. Then multiply by the smaller number and divide by the larger. Thus at the end we have to put down $648 \times \frac{6}{9}$.

* It is frequently the case that the unit line is absurd, even though the ultimate conclusion is sound. This should be pointed out to boys; if mathematics is a training in exact reasoning, boys ought to know when they are making an absurd statement, even if it is convenient to make it.

† It is important not to get on to the shortened form too soon.

‡ $\frac{648 \times 6}{9}$ is really the form the boy would expect and that may be used at first; $648 \times \frac{6}{9}$ leads on better to ratios.

Ex. 2. 200 men coal the ship in 10 hours,

250 „ „ 10 × — hours.

[Here the number of hours has to be decreased; we multiply by the smaller (200) and divide by the larger (250).]

What are the advantages of the second method?

(i) It is shorter.

(ii) It eliminates the unit line which is often absurd—a ship could hardly be coaled by one man.

(iii) It leads to the idea of ratio—changing a number in a certain ratio.

(iv) It prevents mistakes in working questions involving fractions or decimals.

The following examples illustrate (iv) above.

Ex. 3. *If $\frac{1}{3}$ of a ton of lead costs £11, what is the price of $\frac{1}{2}$ ton?*

$\frac{1}{3}$ of a ton costs £11,

$\therefore \frac{1}{2}$ „ £11 × —.

[$\frac{1}{2}$ is more than $\frac{1}{3}$. $\therefore$ $\frac{1}{2}$ a ton will cost more than $\frac{1}{3}$ of a ton, $\therefore$ the £11 must be increased, $\therefore$ multiply by the larger ($\frac{1}{2}$) and divide by the smaller ($\frac{1}{3}$).]

$$\therefore \tfrac{1}{2} \text{ a ton costs } £11 \times \frac{\frac{1}{2}}{\frac{1}{3}} = £11 \times \frac{\frac{1}{2} \times 6}{\frac{1}{3} \times 6} = £11 \times \tfrac{3}{2}.$$

If such a sum is set to be done by Unitary Method, it is common experience to find that half the class get as the second line, "1 ton costs £11 × $\frac{1}{3}$."

I grant that, in such a simple case, the boy is easily convinced of his mistake; with harder fractions it is not so easy; but the mistake is entirely avoided by omitting the unit line.

Later on, in the science laboratory, boys frequently get into difficulties in dealing with Unitary Method questions. Here is one in which the unit step is the last step; the shortened method I have just suggested makes the thought simpler.

Ex. 4. *In a certain experiment* 83·3 *c.c. of hydrogen were liberated by dissolving* 0·0735 *gm. of a metal. How much would be liberated by dissolving* 1 *gm. of the metal? (The volume liberated is proportional to the weight of metal dissolved.)*

0·0735 gm. liberates 83·3 c.c.,

∴ 1 gm. „ 83·3 × — c.c.

[1 is larger than 0·0735, ∴ the volume must be increased, ∴ multiply by the larger (1) and divide by the smaller (0·0735).]

$$\therefore\ 1 \text{ gm. liberates } 83{\cdot}3 \times \frac{1}{0{\cdot}07} \text{ c.c.}$$

Third Method

Ultimately, after the boy is thoroughly familiar with ratio, he would think of the above examples by means of ratios or multiplying factors; thus

Ex. 1. The distance will be decreased in the ratio 6:9, or the multiplying factor is $\frac{6}{9}$.

Ex. 2. The number of hours will be decreased in the ratio 200:250, or the multiplying factor is $\frac{200}{250}$.

Ex. 3. The cost will be increased in the ratio $\frac{1}{2}:\frac{1}{3}$, or the multiplying factor is $\frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}$.

Ex. 4. The volume will be increased in the ratio 1:0·0735, or the multiplying factor is $\frac{1}{0{\cdot}0735}$.

Reference should be made here to "Algebra," chap. IV, p. 169.

CHAPTER V

DECIMALS

Should decimals or vulgar fractions be taken first?

To my mind, it is immaterial. It is absurd to argue that decimals cannot be taken before vulgar fractions on the grounds that a child does not know what $\frac{1}{10}$ is until vulgar fractions have been done. I grant the child must know what a vulgar fraction is, in particular what $\frac{1}{10}$ is, and that $\frac{1}{10}$ of $\frac{1}{10}$ is $\frac{1}{100}$; but that is about all the knowledge that is required before decimals are begun, and it can be illustrated very nicely on squared paper ruled in inches and tenths. To add 0·5 and 0·8 is certainly easier than to add $\frac{1}{2}$ and $\frac{4}{5}$.

The great advantage of the decimal is that it is merely an extension of the ordinary notation, and most of the work involves very few changes of ideas or methods.

To pave the way for decimals it is wise to refresh boys' memories about our ordinary notation.

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Consider 1 2 3. What do the digits represent? Multiply it by 10; 1 2 3 0. What do the digits now represent? Point out that, instead of saying that we have added a "0" to multiply by 10, it is better to say that we have moved all the figures one place to the left (and filled the gap with a "0").

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Again consider 1 1 1 1. Each "1" represents 10 times as much as the "1" to its right, and each "1" represents $\frac{1}{10}$ as much as the "1" to its left.

If we put a 1 to the right of the unit column (we had better put a point in to show which is the unit) the new 1 should still

represent 1/10th of the 1 to its left, i.e. 1/10th of a unit. So we have

Thousands	Hundreds	Tens	Units		Tenths
1	1	1	1	·	1

For a first lesson in decimals I would suggest taking a metre rod divided into tenths—I have used a metre scale which happened to have small holes drilled at every 10 cm., and through these I passed a piece of red wool and tied it round the ruler (the holes were useful in preventing the wool from slipping).

By aid of this, make some measurements; the length of the room, 7 metres and 8 parts; as there are 10 of the parts, each is 1/10th of a metre—this can be written 7·8 metres.

At once **addition** is easy; take cases from actual measurements, say the length of the room, 7·8 metres, and the width of the passage 1·4 metres, add them.

7·8 metres		(We have 12 of the tenths, 10 of
1·4	,,	them make 1 metre, so we
9·2	,,	carry in the ordinary way)

This is easily extended to the next place, and the only thing that is necessary is to get the child always to write the unit figure under the unit figure or, what comes to the same thing, the decimal point under the decimal point.

The reason for the decimal point presents no difficulty.

Subtraction comes at once.

A little practice here can be given just to improve the pace in mere addition and subtraction; and to the child there is the added interest in doing something with an element of novelty in it.

Multiplication and **division** by 10, 100 or 1000 should be dealt with here. Point out that the old rules for multiplying and dividing by 10 or 100 should now be absorbed in the more

general rules of moving the figures up or down so many places —that is better than thinking of moving the decimal point.

Multiplication and **short division** by a single figure (an integer from 2 to 9) are equally simple to explain and are picked up at once.

General multiplication and **division** are much more difficult.

There are several methods in common use; and much unnecessary difficulty is created if a boy learns one method and later on is made to change to another method.

It is a great pity that no single method stands out so clearly as the best that it is universally adopted: this would be a great advantage when a boy changes from one school to another (e g. when he goes from an elementary school to a secondary school, or from a preparatory school to a public school). In any one school, at any rate, one method should be adopted throughout. I have known schools in which one master insisted on one method, and another master on another.

Of course, if pupils come to a secondary or public school with their multiplication and division of decimals perfect, there is no need to bother about what method they use, except that they may not follow when a master has to do a piece of multiplication or division in the course of working a problem on the board. But what is to be done in the case in which new boys at a secondary or public school are not perfect at multiplication and division of decimals? Two courses are open—either to make all such boys adopt the one method the school chooses, or to try to perfect each boy in the method he has learnt before; each school must decide which of the two courses it will adopt; much will depend on the number of boys who are not perfect. Of course the danger of making a boy learn a new method is that he will muddle the two; but, on the other hand, it is impossible to teach boys as a class when two or three different methods are being used.

Of course the moral to be learnt from this is that, if a school starts multiplication and division of decimals, it should at once grind at it so hard that the processes become perfect in the term in which they are begun, so that there is no likelihood of a boy leaving till he is certain of his decimal point. To ensure this I would suggest that addition, subtraction, multiplication by a single integer and division by a single integer should be done one term, and that general multiplication and division should be left to be begun at the beginning of the next term, and should at once have a steady grind so that they become practically perfect in that term

After these two rules have been learnt I would start every arithmetic lesson with one multiplication and one division sum (they should not be long sums), and repeat that at every lesson until every boy succeeds in getting the decimal point right every time.

To my mind, in the following term, a good many lessons might start in the same way; interest can be kept going by making a race of the two sums, and so improving pace. It is surprising how little time this takes if well organised.

MULTIPLICATION OF DECIMALS

Method I*a*

Probably the oldest method is to do the multiplication irrespective of decimal points and then count off from the right of the product as many figures as there are after the decimal point in the two original numbers thus:

4 9·7 2
4 0·4 0 3

The simplest explanation is that the 2 stands for $\frac{2}{10^2}$ and the 3 for $\frac{3}{10^3}$, $\therefore$ their product is $\frac{6}{10^{2+3}} \quad \frac{6}{10^5}$.

Every boy should be familiar with this method for such cases as $0{\cdot}2 \times 0{\cdot}3$ and $(0{\cdot}01)^2$ and $(0{\cdot}004)^2$.

If this method is used, there is considerable danger that it will become a blind rule too soon and the master must constantly refer to the explanation. Of course it has ultimately to become a mechanical rule.

METHOD I*b*

A slight variant of this method is to place the multiplier so that the unit figure of the multiplier is under the last figure of the number to be multiplied, then the multiplication of each figure begins under that figure of the multiplier—thus in the following example:

```
      4 9·7 2
          4 0·4 0 3
1 9 8 8·8
      1 9·8 8 8
          ·1 4 9 1 6
2 0 0 8·8 3 7 1 6
```

The advantage of this method is that it is analogous to what boys have already learnt in multiplying whole numbers.

The disadvantage is that boys are apt to get two decimal points in their answer (e.g. in the above example 2008·83·716) through repeating the decimal point in the multiplier and that in the multiplicand.

METHOD II

Yet another method is to do the multiplication irrespective of the decimal points and to place the decimal point in the answer by making a rough approximation.

This is a very good method in some cases, and one which most boys of mathematical ability will use to some extent at a later stage of their development, e.g. when using a slide-rule; but it is very uncertain with weak boys.

Method III

A third method is that in which the multiplier is changed to standard form (i.e. with one figure to the left of the decimal point). I propose to show fully how I should treat it with a class.

First let us take 83·26 × 2

$$\begin{array}{r} 83{\cdot}26 \\ 2 \\ \hline 166{\cdot}52 \end{array}$$

Lay stress on the fact that when multiplying by 2 units, every figure maintains its own place, and the result of multiplying each figure comes under that figure.

Then 83·26 × 2·4

$$\begin{array}{r} 83{\cdot}26 \\ 2{\cdot}4 \\ \hline 166{\cdot}52 \\ 33{\cdot}304 \\ \hline 199{\cdot}824 \end{array}$$

We have learnt all about the multiplication by the 2. A few questions will elicit the fact that the multiplication by the 4 must start one place to the right; it may be wise to write up above the result of dividing 83·26 by 10 (taking care to keep the decimal points above one another) and then to multiply this by 4. But the great point to bring out is the analogy to ordinary long multiplication.

We have now learnt how to multiply by a number in standard form and much practice must be given before going on to the next stage.

We have now to learn how to multiply 83·26 by 32·4.

First of all we must take examples such as 40 × 30 = 400 × 3. Then we might consider such a problem as finding the area of a rectangle 83·26 yards by 32·4 yards. If we divide the rectangle into 10 strips of equal width and place the strips end to end, we have a rectangle of the same area but 10 times as long and of

1/10th the width. Hence 83·26 × 32·4 = 832·6 × 3·24; which is a type with which we can deal.

Now we want some practice with moving decimal points. I would advocate using arrowheads to show where the decimal point is to go and noticing that the decimal points always move in opposite directions.

567·24 × 26·3 = 5672·4 × 2·63

26·4 × 0·07 = 0·264 × 7·

* 0045·7 × 0·00032 = 0·00157 × 3·2

The complete work for a sum would appear thus:

83·26 × 32·2 = 832·6 × 3·22.

```
  8 3 2·6
      3·2 2
2 4 9 7·8
  1 6 6·5 2
    1 6·6 5 2
2 6 8 0·9 7 2
```

Some people advocate setting it down as follows, but there is the danger of two decimal points appearing as pointed out on p. 109.

```
  8 3 2·6
      3·2 2
2 4 9 7·8
  1 6 6·5 2
    1 6·6 5 2
2 6 8 0·9 7 2
```

* The two 0's to the left of the 4 are put in after the sum has been set down as 45·7 × 0 00032.

WHICH METHOD SHOULD BE ADOPTED FOR MULTIPLICATION?

There is no question that every child should be familiar with Method I for examples such as $(0{\cdot}02)^2$, and also with Method III for making rough approximations.

The disadvantage of Method III is that children get muddled about the moving of the decimal points opposite ways in multiplication and the same way in division; this muddling ought not to occur if division sums are always written down in the form $\frac{783{\cdot}21}{37{\cdot}2}$, but the fact remains that the muddling does occur.

On the whole I believe that the safest plan is to adopt Method I; though I must confess that, if I had the entire teaching of a child, I should adopt Method III and use Method I as a check.

DIVISION OF DECIMALS

For division there are three methods:

I. To change the divisor into a whole number.

II. To change the divisor into standard form, i.e. with one figure to the left of the decimal point.

III. To divide irrespective of decimal points and to place the decimal point by a rough approximation.

Method III needs no further explanation.

I will give an example of Methods I and II.

In both cases I would advocate that the sum should never be copied on to the boys' paper in the form $0{\cdot}00724 \div 0{\cdot}0892$ but always as a fraction, thus $\frac{0{\cdot}00724}{0{\cdot}0892}$, with the decimal points under one another.

The only explanation that is necessary then is that the value of a fraction is unaltered by multiplying (or dividing) the numerator and denominator by the same number, a power of 10 in this case*.

* If vulgar fractions have not been done before, this must be treated fully now.

If the divisor is changed to a whole number, the work is exactly like ordinary long division, thus

$$\frac{*0{\cdot}0072|4}{0{\cdot}0892|} = \frac{72{\cdot}4}{892{\cdot}} \qquad \begin{array}{r} 0{\cdot}081 \\ 892\overline{)72{\cdot}40} \\ \underline{71{\cdot}36} \\ 1{\cdot}040 \\ 892 \end{array}$$

If the divisor is changed to standard form, i e., so that it has one figure to the left of the decimal point, the work is as follows:

$$\frac{0{\cdot}00|724}{0{\cdot}08|92} = \frac{0{\cdot}724}{8{\cdot}92} \qquad \begin{array}{r} 0{\cdot}081 \\ 8{\cdot}92\overline{)0{\cdot}7240} \text{ etc.} \\ \underline{7136} \\ 1040 \\ 892 \end{array}$$

The explanation is that the work is like dividing by 8 and a bit. If you divide by 8 by short division, the first figure comes under the 2, but we place it above so as to be out of the way. That settles the place of the first figure in the answer, and all that is necessary is to count off where to place the 8 times the divisor.

In practice this method is easily understood and avoids mistakes in the decimal point, *provided the analogy to short division is kept to the fore*. Always insist on the boy saying "I am dividing by 8 and a bit" (or whatever the number is and a bit). The only difficulty is in a case like the following:

$$8{\cdot}92\overline{)83{\cdot}26};$$

the boy sees that by short division by 8 he gets a 1 over the 8, he finds that the first figure should be a 9, but he forgets sometimes that it should go over the 3.

* It is helpful to put in a line to indicate where the decimal points are to be moved to.

Of these two methods I would recommend the second; it is perfectly easy to teach, it is the method used for making a rough approximation, and is the better in case of contraction later on.

It should be pointed out that in both methods the remainder has had its decimal point altered, but the remainder is seldom wanted.

RECURRING DECIMALS

At the end of the last century much time was devoted to recurring decimals and their conversion into vulgar fractions. Happily that work has all disappeared from our schools. All that a child need know nowadays is that

$$\tfrac{1}{3} = 0{\cdot}3333 \ldots \text{ going on for ever,}$$

and that we call this a recurring decimal and represent it by $0{\cdot}\dot{3}$; also that $\frac{2}{3} = 0{\cdot}\dot{6}$ and $0{\cdot}\dot{9} = 1$.

DECIMALISATION OF £ *s.* *d.*

Every child ought to be able to express any sum in shillings and pence as a decimal of £1, correct to any given number of places, by the simple method of first expressing the pence as a decimal of 1 shilling, and then expressing the total number of shillings as a decimal of £1.

$\frac{1}{4}d = {\cdot}25d.$, $\frac{1}{2}d. = {\cdot}5d.$, $\frac{3}{4}d. = {\cdot}75d.$ should be obvious.

Express 17*s*. $8\frac{1}{4}d$. *as a decimal of* £1 *correct to* 4 *places of decimals.*

At first some teachers may prefer to arrange the work thus:

$$\begin{aligned} 17s.\ 8\tfrac{1}{4}d. &= 1\ 7s.\ 8{\cdot}2\ 5d. \\ &= 1\ 7{\cdot}6\ 8\ 7\ 5s. \\ &= £\ 0{\cdot}8\ 8\ 4\ 4. \end{aligned}$$

Later the following arrangement is best:

$$\begin{array}{r|l} 12 & 8{\cdot}25\,d. \\ \hline 20 & 17{\cdot}6875\,s. \\ \hline & £0{\cdot}8844 \end{array}$$

(the 17*s*. are put in after the division by 12)
(divide by 2 and move all the figures one place to the right)

To express a decimal of £1 *in shillings and pence.*

First point out that 1 farthing is £$\frac{1}{960}$ = £0·001 approximately.

Hence, if the result is wanted within a penny, only the first three figures after the decimal point need be considered.

If the result is wanted within a farthing, it will be wise to retain four figures.

Express £14·3257831 *in* £ *s. d. correct within a farthing.*

We need only keep 4 decimal places, but we correct the last figure.

At first the work may be arranged thus:

£14·3258 = £14. 6·516*s*.
= £14. 6*s*. 6·192*d*.
= £14. 6*s*. 6*d*. 0·768 farthing
= £14. 6*s*. $6\frac{1}{4}d$.

Later it may be arranged thus:

$$\begin{array}{l} £14\,|\,{\cdot}3258 \\ \quad 20 \\ \hline \quad 6\,|\,{\cdot}516\,s. \\ \quad 12 \\ \hline \quad 6\,|\,{\cdot}192\,d. \\ \quad 4 \\ \hline \quad\ {\cdot}768 \text{ farthing} \end{array}$$

(multiply by 2 and move all figures one place to the left)

£14. 6*s*. $6\frac{1}{4}d$.

Every child should be familiar with the above, and should realise that the processes are general and can be applied to tons, cwts., quarters or yards, feet, inches, etc.

CHAPTER VI

VULGAR FRACTIONS

All through the formal work with fractions I would urge the teacher to have in mind that, later on, the teaching of fractions in algebra will be a simple easy piece of work, *provided vulgar fractions are taught properly in arithmetic**, but that, if the arithmetic teaching is bad—a mere collection of rules—the teaching of fractions in algebra will take much longer.

At a comparatively early age children have quite a clear idea as to the meaning of one-half; the meanings of one-third, one-quarter, two-thirds and three-quarters follow quite quickly. All these come long before proper lessons have begun and that is the basis on which the teacher should build In the kindergarten or nursery stage this knowledge should have been consolidated, so that the child realises that $\frac{3}{5}$ths of a cake is found by dividing the cake into 5 parts *all of the same size* and taking 3 of those parts. Much delightful work with paper folding and various apparatus can be done in this stage; but I want to consider a later stage.

The first point to bring out in the teaching of vulgar fractions is that

The value of a fraction is unaltered by multiplying (or dividing) numerator and denominator by the same number.

I would call this the golden rule about fractions; but it should not be fired at the children in such a form. At the start I would avoid the words "numerator" and "denominator"—

* It is quite true that many of the explanations which the arithmetic teacher gives will be apparently forgotten. Nevertheless, they will stay in the pupil's subconscious mind, and, when algebra fractions are being done, the good teaching of vulgar fractions will bear its fruit, in fact many things that have been in the child's subconscious mind will then come into the conscious plane and remain there.

let them creep in gradually—speak of the top and bottom of the fraction. Take practical examples.

Cut a cake into halves and quarters.

At once it is clear that $\frac{1}{2} = \frac{2}{4}$.

Again 6*d.* is half of 1*s.*, and 1*s.* = 12*d.* $\therefore \frac{1}{2} = \frac{6}{12}$.

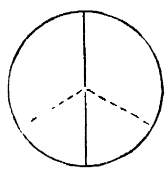
From this and similar cases we infer that, if we multiply the top and bottom of a fraction by the same number, the fraction still has the same value.

Here will come some examples on reducing fractions; but there is not the slightest need to mention H.C.F.; divide top and bottom by any number that will divide both. The numbers involved should be simple, there is not the slightest need to deal with fractions such as $\frac{313}{399}$. Examples such as "fill up the gaps in $\frac{1}{2} = \frac{\ }{16}$, $\frac{2}{5} = \frac{\ }{25}$" will help with addition directly; they lead on to finding x in cases such as $\frac{5}{10} = \frac{x}{2}$, thus paving the way for algebra later. I should not hesitate to ask a bright class to simplify $\dfrac{2\frac{1}{2}}{3\frac{1}{2}}$; I can imagine the strict logician saying "But such a fraction has not been defined," but I would answer "What does that matter to the child?"

ADDITION

Let us try to add $\frac{1}{2}$ and $\frac{1}{3}$.

Take a round cake; cut it into halves, and also into thirds. Now then—how much do one-half and one-third of the cake make?



Ask for suggestions (possibly in vain). Cut the cake into sixths; at once the half is three-sixths, the third is two-sixths and together they make five-sixths—"Count them."

$$\therefore \frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}\text{*}.$$

* A very able teacher of another subject once told me that in his young teaching days he was doomed to teach arithmetic in a preparatory school: he saw his headmaster give the lesson which I have just sketched, finishing with "Now eat the cake." He told me of the impression it made on his mind, and he added "I had never understood before what you do when you add $\frac{1}{2}$ and $\frac{1}{3}$."

Such a lesson to the mathematically-minded may seem waste of time, but it is not; it makes an impression on the child's mind.

But the lesson is not done with; we have still to pick out the fundamental ideas. I should like to lay stress on the importance of following up the concrete illustration with some discussion of the fundamental principles involved. After the concrete example it is a good thing to attempt an abstract case at once; but this does not mean that concrete examples will not be wanted again

Take another case: add $\frac{3}{4}$ and $\frac{2}{3}$*.

What must we do?

What did we do with one-half and one-third?

We changed to fractions with the same denominator†, and that denominator had to be divisible by 2 and also by 3.

What must we do in this case? Someone will suggest the right thing—we must change each fraction to one with a new denominator divisible by 3 and also by 4. Obviously 12 will do.

Now $\frac{3}{4} = \frac{}{12}$. What did we do to the 4 to get 12? Multiplied by 3. Then we must do the same to the numerator.

Therefore $\frac{3}{4} = \frac{3\times 3}{12} = \frac{9}{12}$.

Similarly $\frac{2}{3} = \frac{2\times 4}{12} = \frac{8}{12}$.

Therefore $\frac{3}{4} + \frac{2}{3} = \frac{9}{12} + \frac{8}{12} = \frac{9+8}{12} = \frac{17}{12} = 1\frac{5}{12}$.

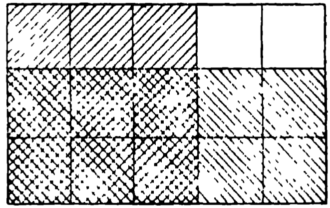
Take another case: $\frac{1}{6} + \frac{3}{8}$. Here we shall have 48 suggested as a denominator; but 24 would do, and would be better because it is smaller.

We always choose the smallest denominator that will do. No need for such technical terms as L.C.M. I would avoid them for a long time, even till long after algebra has been begun.

* $\frac{3}{4} + \frac{2}{3}$ can be illustrated nicely by taking three-quarters of a shilling and two-thirds of a shilling.

† I have slipped into the technical term; each master must use his judgment as to how soon he does that: at this stage he will probably still call it the bottom of the fraction.

Many nice illustrations* can be made by means of squared paper or by paper folding. Thus, to illustrate $\frac{2}{3} + \frac{3}{5}$ take a sheet of paper and fold it into thirds one way and into fifths the other, or on squared paper mark out a rectangle 3 by 5. Now shade in two of the thirds and three of the fifths, shading in opposite directions.



(i) There are 15 (= 5×3) small parts.
(ii) $\frac{2}{3}$ of the whole contains 10 of those parts.
(iii) $\frac{3}{5}$ 9
(iv) $\frac{2}{3} + \frac{3}{5}$ 10 + 9
= 19 of those parts.
= the whole plus 4 of those parts.

Or in symbols

$\frac{2}{3} = \frac{10}{15}$, (iii) $\frac{3}{5} =$

(iv) $\frac{2}{3} + \frac{3}{5} = \frac{10}{15} + \frac{9}{15} = \frac{10+9}{15} = \frac{19}{15} = 1 + \frac{4}{15}$.

Now the children are ready for some written work in addition and subtraction of fractions.

Most modern books have examples on addition and subtraction of fractions that deal only with fractions with reasonable denominators. Choose a few easy ones and let them tackle those. Then discuss *vivâ voce* what common denominators will be necessary in the further questions in the book.

E.g. G. and P. *Arithmetic*, Ex. VI *l*, No. 7. Simplify $\frac{4}{15} + \frac{3}{10}$. Denominators $15 \times 10 = 150$ would do, but we can get something simpler, viz. 30.

More written work.

* The wise teacher will not use all those illustrations at once before attempting any written work. Probably he will reserve the squared paper work for the beginning of the next lesson after some written work has been

A little later, we come across harder cases*; thus, in No. 36 the denominators are 10, 42 and 105. Split them up into prime factors:

$10 = 5 \times 2$, $42 = 2 \times 21 = 2 \times 3 \times 7$, $105 = 5 \times 21 = 5 \times 3 \times 7$.

Our common denominator must be $2 \times 3 \times 5 \times 7$ ... and so on.

At first each fraction should be put over the desired denominator.

Thus in $\frac{8}{9} + \frac{5}{6} + \frac{11}{12} = \frac{32}{36} + \frac{30}{36} + \frac{33}{36}$. Having settled that 36 is the required denominator, ask "By what must 9 be multiplied to make 36?" Answer "4." "Then the 8 must be multiplied by 4," etc.

In addition and subtraction of fractions children should always be taught to deal with the whole numbers first†.

$$+ \frac{1}{4}$$
$$= 4 + \frac{8}{20} + \frac{5}{20} + \frac{14}{20}$$
$$= 4 + \frac{8+5+14}{20}$$
$$= 4 + \frac{27}{20}$$
$$= 4 + 1\frac{7}{20}$$
$$= 5\frac{7}{20}.$$

Above I have illustrated very carefully the line of thought for the pupil to go through. I cannot condemn too strongly the sort of blind rule many boys seem to have learnt from somewhere; thus, in the above example it is quite common to hear this sort of thing:

"Find the L.C.M." (They don't say of what.) "It is 20."

Then $\frac{2}{5} + \frac{1}{4} + \frac{7}{10} = \frac{}{20}$.

"5 goes into 20 four times, so I multiply the top by 4; 4 goes into 20 five times, so I multiply the top by 5, etc.," but invariably a pupil who gives that as an explanation has not the foggiest

* Possibly too hard yet; it may be well to postpone them.

† I mention this particularly because it is quite common to find the following:

$$1\frac{2}{5} + 3\frac{1}{4} + \frac{7}{10} = \frac{7}{5} + \frac{13}{4} + \frac{7}{10} = \text{......, etc.}$$

Whether some teachers teach this or not I cannot say, but it is so common that I have my suspicions.

notion of why he does it, and in very many cases I have my suspicions that he never has understood it, but has been taught a mere blind rule. It is just as easy to say "5 must be multiplied by 4 to give 20, therefore I must multiply the top by 4"—this keeps the fundamental principle in view and helps with other work later.

SUBTRACTION

In subtraction all is plain sailing except for the one difficulty illustrated by the following:

$$\begin{aligned} 4\tfrac{5}{12} - 1\tfrac{1}{2} = 3 + \tfrac{5}{12} - \tfrac{6}{12} &= 3 + \tfrac{5-6}{12} \\ &= 2 + \tfrac{12}{12} + \tfrac{5-6}{12}\,{}^{*} \\ &= 2 + \tfrac{12+5-6}{12} \\ &= 2 + \tfrac{11}{12}. \end{aligned}$$

MULTIPLICATION

The actual multiplication of fractions does not present great difficulties; but there are underlying principles which need discussion†.

First of all, what do we mean by the sign $\times$?

Quite definitely it means "multiplied by." But as the children know that $2 \times 3 = 3 \times 2$, it is quite legitimate to think of $\times$ as meaning "times." Here we had better introduce the word "of."

$\frac{1}{2}$ of 8 inches = 4 inches.

3 lengths of 8 inches = 8×3 inches.

Hence we see that $8 \times \frac{1}{2} = \frac{1}{2}$ of 8.

But $8 \times \frac{1}{2}$ we assume to be equal to $\frac{1}{2} \times 8$.

Thus $8 \times \frac{1}{2}$, $\frac{1}{2} \times 8$ and $\frac{1}{2}$ of 8 are all equivalent.

* This step is easily explained.

† It is a mistake to cut this discussion and merely give a mechanical rule. On the other hand, it is a mistake to harp too much on the principles; the mechanical rule will be picked up quickly, let the children use it and discuss the principles again when revising a year or so later—that second discussion will be the more fruitful.

It is an easy extension to see that

$$2\tfrac{2}{3} \times 3\tfrac{3}{4},\ 3\tfrac{3}{4} \times 2\tfrac{2}{3},\ 3\tfrac{3}{4} \text{ of } 2\tfrac{2}{3},\ 2\tfrac{2}{3} \text{ of } 3\tfrac{3}{4}$$

are all equivalent*.

The above discussion should not be laboured; just make the idea familiar.

Then we have to tackle the difficulty that 7 multiplied by $\frac{1}{2}$ gives a result which is less than 7. It is not hard to point out that we are extending the meaning of the word "multiply," just as we extend the meaning of the word "sail" when we talk of a steamer sailing. See G. and P. *Arithmetic*, p. 130.

Now for the actual teaching of the process.

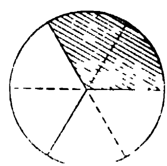
Start with multiplication by a whole number, thus:

$$\tfrac{2}{5} \times 2 \text{ means } \tfrac{2}{5} + \tfrac{2}{5} = \tfrac{2+2}{5} = \tfrac{4}{5}.$$

Now consider $6 \times \frac{1}{2} = \frac{1}{2} \times 6 = \frac{1}{2}$ of $6 = 3$.

Now $\frac{4}{5} \times \frac{1}{2} = \frac{1}{2} \times \frac{4}{5} = \frac{1}{2}$ of $\frac{4}{5} = \frac{2}{5}$.

Again, $\frac{1}{3} \times \frac{1}{2} = \frac{1}{2}$ of $\frac{1}{3}$. Bring in the cake, divide it into thirds and halve each of the thirds.



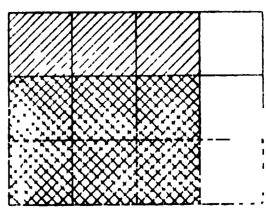
Into how many equal parts is the cake now divided? 6.

What fraction of the whole cake is each part? $\frac{1}{6}$.

In the figure $\frac{1}{3}$ of the cake has been shaded; $\frac{1}{2}$ of the shaded part is what fraction of the whole cake? $\frac{1}{6}$.

Hence $\frac{1}{3} \times \frac{1}{2} = \frac{1}{2}$ of $\frac{1}{3} = \frac{1}{6}$.

Again consider $\frac{3}{4} \times \frac{2}{3}$, i.e. $\frac{2}{3}$ of $\frac{3}{4}$.



On squared paper take a rectangle 3×4 to represent the whole. Divide it into quarters as in the figure and shade 3 of the quarters.

Now draw lines as in the figure to divide the figure into thirds and shade (the other way) $\frac{2}{3}$ of the part already shaded.

* I can imagine the logician quarrelling with this, but it is absurd to worry the child with the logic of such a point.

We see at once that $\frac{2}{3}$ of $\frac{3}{4}$ of the whole consists of 2×3 small parts.

How many small parts make up the whole? 3×4.

$\therefore \frac{2}{3}$ of $\frac{3}{4} = \frac{2 \times 3}{3 \times 4}$.

From this it is easy to deduce the rule.

Of course children should be taken through several examples like the above, but they should not be expected to reproduce the argument for years yet.

DIVISION

Division is more difficult. I will suggest two methods of approach.

First Method

If the children are clear that $5 \div 8 = \frac{5}{8}$, it is natural to assume that $\frac{3}{4} \div \frac{2}{5} = \frac{\frac{3}{4}}{\frac{2}{5}}$*.

This fraction has four "stories" and we only want two. Obviously we should improve matters by multiplying top and bottom by 4 and also by 5†.

$$\therefore \frac{3}{4} \div \frac{2}{5} = \frac{\frac{3}{4}}{\frac{2}{5}} = \frac{\frac{3}{4} \times 4 \times 5}{\frac{2}{5} \times 4 \times 5} = \frac{3 \times 5}{2 \times 4}, \text{ etc.}$$

The rule "Turn the denominator upside down and multiply by that" seems to me quite unnecessary and certainly dangerous for the present. Many pupils will spot it and may then use it.

Skyscrapers such as $\dfrac{1}{\frac{1}{2} + \frac{1}{3} + \frac{\frac{1}{6}}{\frac{3}{4} - \frac{1}{3}}}$ should play no part in elementary arithmetic. There is no need for anything more com-

* I realise the logical difficulty, but once more, what does that matter to the child?

If the child sees the difficulty and feels it, the only thing is to fall back on the second method.

† This should be arrived at by cross-questioning.

plicated than $\dfrac{4\frac{2}{9} - 2\frac{1}{4}}{3\frac{3}{4} - 3\frac{1}{3} + \frac{11}{12}}$, and even such as this are best postponed.

(i) I should like to point out that, in the treatment I have suggested, the whole argument is made to hang on the one rule, "the golden rule about fractions" (see p. 116).

(ii) That exactly the same principles are used when dealing with fractions in algebra.

Second Method

$\frac{3}{4} \div \frac{2}{5}$. Express each of these fractions as fractions with the same denominator. That denominator must obviously be 20.

Now $\frac{3}{4} = \frac{3\times5}{20}$ and $\frac{2}{5} = \frac{2\times4}{20}$.

Thus $\frac{3}{4} \div \frac{2}{5} = \frac{3\times5}{20} \div$

$= (3 \times 5)$ of a twentieth $\div$ (2×4) of a twentieth

$= \frac{3\times5}{2\times4}$.

This can be made concrete* by making the question "How many times is £$\frac{2}{5}$ contained in £$\frac{3}{4}$?

£$\frac{2}{5}$ = 2×4 shillings, and £$\frac{3}{4}$ = 3×5 shillings.

So the question now becomes "How many times is 8 shillings contained in 15 shillings?"

I have only dealt in this chapter with the manipulation of vulgar fractions, but it is essential that the pupil should be constantly dealing with problems involving fractions.

* The wise teacher would take the concrete case first.

CHAPTER VII

AREAS AND VOLUMES

AREA

It is very surprising to find how vague children are in their ideas about area. I once heard a teacher ask a boy—"Do you know anything about area?" The boy's answer was "It is length multiplied by breadth"; and that answer is typical of many boys' ideas about area. Some years ago I examined in arithmetic a large number of future teachers; their efforts at explaining volume showed me that many of them did not understand anything about area or volume—they had merely got a few rules which they could not explain to a class. The above must be my excuse for dealing rather fully with the elements of a subject on which it is fatally easy to get a class to work examples without much understanding; more attention paid to the fundamental ideas would make a great difference to much later work.

First of all, what do we mean by area? No form of words that I know of will make the idea simpler; such statements as "The area of a rectangle is the space enclosed by its sides" are thoroughly pernicious.

It is quite wrong teaching to start with a definition.

Well, having failed to get a definition that is intelligible, what can we do about area? Can we find the area of the blackboard? Let us go back. If we want to find the length* of a room, what do we do? The primitive way to do it is to see how many times it contains the length of one's foot; but let us express this in a different way—(1) we choose some definite

* It is nearly as difficult to find a satisfactory and useful definition of length as it is to find a definition of area.

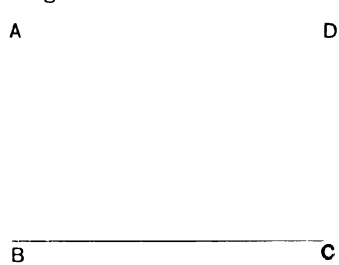
length as our unit, (ii) we see how many times that unit is contained in the length of the room.

Now let us go back to the area of our blackboard. (i) We must choose some definite area as our unit, (ii) we must see how many times that unit is contained in the area of the blackboard. I would suggest as our first primitive unit the area of the duster; see how many times that can be fitted on the board. I know many teachers may scoff at this, but it gives the fundamental idea of measuring an area, which is worth stressing.

Now comes an opportunity of discussing the necessity for a fixed standard of length—a man's foot is not good enough; why? In the same way a duster is not a good enough standard measure for area, for dusters are not all of the same area. Hence it is easy to lead a class on to see that a square foot is a better unit of area.

We now have a proper standard unit of area; cut out a square foot of cardboard and see how many times it can be fitted on to the board. Probably it will be better to draw a rectangle on the board, say 5 ft. by 3 ft., and find its area. Mark the rectangle off into square feet, by aid of the standard cardboard square foot, and count how many square feet there are in the rectangle.

Now let us shorten our counting. In each row we have 5 sq. ft. (why 5? because BC is 5 ft. long, and one square foot stands on each linear foot in BC).



Again, how many rows are there? Three (because there are as many rows as there are feet in AB; for each linear foot in AB there is one row).

Therefore the whole area is 3 times 5 sq. ft., i.e. 15 sq. ft.

I have deliberately put part of the argument in brackets,

because that part should not necessarily be taken with a class the first time they go through the argument.

In this way, after doing some examples, it is not hard to lead a class to the rule:

"To find the number of square feet in the area of a rectangle, multiply the number of feet in the length by the number of feet in the breadth, the resulting number is the number of square feet in the area."

N.B. In this the multiplication is only the multiplication of two numbers, it is not the multiplication of one length by another.

It may be explained that the above statement is often shortened down to "area of a rectangle = length multiplied by breadth," but it must be clearly understood that it is mere shorthand for the longer statement above.

The proofs given above only apply to integers. At a later stage we must deal with fractions or decimals; all that is necessary is to choose a smaller unit: e.g. to find the area of a rectangle whose sides are 1·2 cm. and 1·5 cm., the sides are 12 mm. and 15 mm., so the area is

$$12 \times 15 \text{ sq. mm.} = \tfrac{12 \times 15}{100} \text{ sq. cm.} = 1{\cdot}2 \times 1{\cdot}5 \text{ sq cm.}$$

(See G. and P. *Arithmetic*, § 120.)

Many examples will have to be worked. When lengths are given in feet and inches, sometimes the lengths should be expressed in feet and fractions of a foot, and sometimes in inches; both methods must be familiar and much judgment will be necessary to decide which is the better method in a particular case.

The areas of triangles, trapezia, etc., I shall consider later (see "Geometry," chap. IX, p. 275).

VOLUME

To find the volume or capacity of a solid, the first thing we have to do is to develop the idea of volume just as we developed the

idea of area. We can go through the same sort of steps—(i) choose a unit of volume, (ii) see how many times it is contained in the required volume. Suppose we want to find how much water a jug will hold, and suppose we take a tumblerful of water as our unit of volume.

Then we get on to the necessity for fixed standards as units of volume. We may take a pint or a cubic inch as our unit of volume; we had better have both, for different purposes.

Now to find the volume of a rectangular block.

In very many cases the argument given to a class practically amounts to this: "area = length multiplied by breadth, ∴ volume = length multiplied by breadth multiplied by thickness." This is bad.

It is not difficult to make a class see the fundamental idea, though it is unlikely that they will be able to reproduce the argument.

Take a rectangular block, say 9 in. by 4 in. by 3 in.

Divide its base up into square inches* and mark it off into layers each 1 in. thick.

In the bottom layer, on each square inch, there stands one cubic inch—therefore, there are as many cubic inches in the bottom layer as there are square inches in the area of the base; and the same is true of each layer.

Hence we can lead up to the rule:

"To find the number of cubic inches in a solid of uniform cross-section, multiply the number of square inches in the cross-section by the number of inches in the length."

N.B. Again we are multiplying numbers only.

It will be noted that I have stated the rule in a form which applies not only to a rectangular block but also to any solid of uniform cross-section. It is quite unnecessary to have a separate rule that applies only to a rectangular block.

* Possibly by pasting a piece of inch paper on to it.

I have not put the above argument out very fully. If it is done at an early age, no doubt it will be wise to build up the solid with inch cubes and then to count them. If it is done at a later age, it is possible to do it without handling the concrete cubes, but the fundamental idea is the same in each case.

May I make a plea for the distinction between "volume" and "capacity"? It is a help to boys if the teacher uses these words correctly.

There is no need to refer here to the text-book examples.

CHAPTER VIII

TABLES AND THE METRIC SYSTEM

It is unnecessary I hope to plead today for the simplification of tables. Troy weight is dead as a table to be used in school, but many teachers must have suffered under it when children.

The following are the only British tables of weights and measures that should be introduced into ordinary arithmetic. If any other tables are used, the boy should be supplied with the table for reference when he does the sum.

BRITISH

Length.

12 inches = 1 foot
3 feet = 1 yard
22 yards = 1 chain
10 chains = 1 furlong
8 furlongs = 1760 yards = 1 mile

Area.

12^2 or 144 square inches = 1 square foot*
3^2 or 9 square feet = 1 square yard
22^2 or 484 square yards = 1 square chain
10 square chains = 4840 square yards = 1 acre
640 acres = 1 square mile

Volume.

12^3 or 1728 cubic inches = 1 cubic foot*
3^3 or 27 cubic feet = 1 cubic yard

* Many boys do not seem to realise that 12 in. = 1 ft., $\therefore$ 12^2 sq. in. = 1 sq. ft. and 12^3 cub. in. = 1 cub. ft.

Weight.	16 ounces = 1 pound
	14 pounds = 1 stone
	28 pounds = 1 quarter
	4 quarters = 8 stone = 112 pounds = 1 hundredweight (cwt.)
	20 cwt. = 1 ton
Capacity.	2 pints = 1 quart
	4 quarts = 1 gallon

FOREIGN

Metre, gram, litre with the usual prefixes, 1 litre = 1000 c.c., franc and centime, dollar and cent.

Sums ranging from ounces to tons, or from inches to miles, should not be set. In general no body which weighs more than a few pounds is weighed within an ounce In the same way a distance of a mile is seldom measured to a greater degree of accuracy than the nearest yard.

METRIC SYSTEM

Why is it that some boys use vulgar fractions when working in the metric system? I can understand a boy doing it once, but he ought to get such a wigging for it that he will never do it again. The whole value of the metric system is that it lends itself to decimal work.

In introducing a boy to the metric system it is a mistake to introduce all the denominations at once. For example, in introducing length the metre and centimetre might be familiarised first, and then perhaps the kilometre; when these are thoroughly familiar, it is of interest to the boy to learn the other names, and to be introduced to the corresponding measures of weight.

I suppose boys always will make mistakes in changing square centimetres into square metres.

They need constantly reminding that 100 cm. = 1 m.,

$$\therefore\ 100^2 \text{ sq. cm.} = 1 \text{ sq. m.}$$

ADDITIONAL FACTS ABOUT TABLES

The following are useful:

1 km. = $\frac{5}{8}$ mile very nearly.
1 m. = 3 ft. $3\frac{3}{8}$ in. very nearly.
1 c.c. of water weighs about 1 gm.*
1 cu. ft. of water weighs about 1000 oz. or about 62·3 lb.*
1 gallon of water weighs about 10 lb.*
1 cricket pitch = 1 chain = 22 yd.
10 square cricket pitches = 1 acre.

To familiarise the class with a metre it is useful to have a length of 1 m. painted on the wall and labelled.

FRACTIONS OF £1

Every child should be familiar with the following:

$\frac{1}{3}$ of £1 = 6*s*. 8*d*.
$\frac{2}{3}$ of £1 = 13*s*. 4*d*.
$\frac{1}{8}$ of £1 = 2*s*. 6*d*. = half a crown

and $\frac{2}{8}$, $\frac{}{8}$, of £1 = 2 half crowns (5*s*.), 3 half crowns (7*s*. 6*d*.), ...

* These depend on the temperature of the water to some small extent.

CHAPTER IX

SQUARE ROOT

Very many years ago I looked over the arithmetic examination papers of a large number of schools; there was a square root question in the paper, and the school that made most mistakes in that question was a school in which all the candidates tried to explain what they were doing.

My own recollection is that I learnt square root simply as a rule. Possibly it was explained to me; but if so, the explanation made no impression whatever; years afterwards I came to square root in algebra, and then, without any help, the reasoning for the rule both in algebra and arithmetic became quite clear.

The explanation by algebra is undoubtedly the one that will appeal to the mathematician and the one that can be made complete, but if that were the only explanation available I am afraid that I should say that square root would be the one exception to my creed of never giving a child a blind rule; I should be tempted to give him the blind rule and then hope to get him to see some part of the rhyme and reason after he had done square root in algebra. However, we need not resort to that nowadays. Prof. Nunn has suggested a delightful geometrical approach to the subject which will make at least part of the rule intelligible and reasonable to the child, but even with this it will still be a rule, I fear.

Consider the following problem.

If we are given the number of square yards in the area of a square field, what is the number of yards in its length?

First of all we will take a case in which the result is obvious, but we will deal with it in a way that leads up to the general method.

Let us suppose the given area is 121 sq. yd.

We know that $10^2 = 100$. In the figure the area of the large square is 121 sq. yd. and we have cut off a square whose area is 100 sq. yd. The shaded area on the right has been moved up to the left.

Now $121 - 100 = 21$,

$\therefore$ the rectangle **ABCD** has an area of 21 sq. yd.

If the side of the whole square is $10 + x$ yd.

$$\text{AD} = 2 \times 10 + x \text{ and } \text{AB} = x.$$

$$(2 \times 10 + x)\,x = 21,$$

$$x = \frac{21}{2 \times 10 + x}.$$

Now we know that x is small compared with 2×10,

$$\therefore\ x = \frac{21}{2 \times 10} \text{ approx.} = 1 \text{ approx.}$$

And it is obvious $x = 1$ is a solution.

Now let us put this out as a sum.

First of all box off the digits in pairs starting from the decimal point (it is quite easy to show the reason for this by comparing 2^2 and 20^2, 9^2 and 90^2, etc.)

```
      1|    |
      1| 2 1|
      1 0 0
2 )     2 1
```

Take away the 100; and, if you look back, we divided the 21 by $2 \times 10 + x$ where x is the number of times it goes.

So the whole sum is

```
      1| 1  |
      1|2 1 |
      1 0 0
  21 ) 2 1
       2 1
```

Now take another example. Suppose the area is 529 sq. yds.

$$20^2 = 400 \text{ and } 30^2 = 900.$$

Obviously the side of the square is twenty something.

Again refer to the figure, the large square now represents 529 sq. yd. and the smaller one represents 400 sq. yd.

The remainder, 129, shows us the number of square yards in **ABCD.**

```
      2|    |
      5|2 9 |
      4 0 0
  4  ) 1 2 9
```

AD $= 2 \times 20$ plus a bit.

I.e. twice the answer we have found plus a bit.

Now forty something will go about three times. So put in 43 as a divisor and it goes exactly three times.

It is far easier to explain all this on the board than it is to write out the explanation here. Each time draw a figure and mark in various areas and lengths.

Next take an example like, given area $= 5{\cdot}45$ sq. in.

```
         1 5
       4·
  43 ) 1· 4 5
       1· 2 9
 463 )   1 6 0 0
         1 3 8 9
```

We get the 2 as before; then forty something will go ·3 times; point out at this stage that we have subtracted $(2{\cdot}3)^2$ from the whole area, this can be done arithmetically by checking that $4 + 1{\cdot}29 = 2{\cdot}3^2$, also by geometry; so that we have just to repeat the process.

So much explanation I would give; whether I should pursue it farther and how much I should harp on it would depend on the brightness of the class. I should certainly not expect them to be able to reproduce the explanation, but some of it would stick and in revision later they would grasp more of it.

The class will need some drill at it; a little concentrated drill at first and then an example a day for some weeks.

CHAPTER X

THE FIFTH RULE IN ARITHMETIC. COMMON SENSE

So far I have dealt almost entirely with the mechanical processes of arithmetic, though the child will have applied them to many simple problems. Before I go on to consider some of their many applications, I want to emphasise the fact that the wise teacher should aim at getting the child to use common sense and acquire power, and that he will defeat his own end by merely teaching rules.

It is really surprising how the premature giving of a rule destroys the child's power of using common sense. It is common knowledge to most people that a price of *3d.* per article is equivalent to *3s.* a dozen; but it is very surprising to me to find how many boys (and even grown-ups) say that they know that it is so, but they do not understand why it is. Ask a well taught boy to explain the rule and he will probably say "If 1 article costs *3d.*, 12 cost 3 × *12d.*, which is *3s.*"; but how many will say "If 1 article cost *3d.*, 12 cost *36d.*, which is *3s.*" To my mind, there is a world of difference between the two explanations; the boy who gives it in the first form shows much more mathematical power than the other—he has more power of generalisation. Still, the boy who gives the second answer is far ahead of the boy who regards it as a wonderful trick, instead of mere common sense.

I do not want my readers to think that I would attempt to avoid all rules; but the number of them should be as small as possible, and they should be drawn out of the child instead of being thrown at his head.

In general, the best way to tackle any new idea or type of example is to discuss in class a certain number of questions in

which the arithmetical computation is so simple that it can be done in the head. This will develop the underlying principles, and children will then be able to do similar questions involving more computation. Often a child will see how to get the required result, but will not necessarily see how to express the reasoning in an intelligible form. This question of form is important. I would advocate that directly the child grasps the process from examples involving only mental computation, he should try to write out such examples (preferably those that have been discussed in class) in the best form he can; the form and style of these should be discussed and criticised before going on to examples involving more computation. It is a mistake to let the child adopt a poor arrangement of the argument, trusting to improving it later*; and it is a mistake to be thrashing out the arrangement of the argument in a sum in which there is the distraction of heavy computation.

It should be recognised by teachers that there is a distinct difference between the ability to see how to get a result and the ability to put out the argument in an intelligible form. There are teachers who argue that so long as the child sees how to get the result, that is all that matters; but the other ability is worth having, it helps to clarify the child's thought, and in long complicated sums it is necessary to put out various steps in the argument, not only for the sake of whoever looks over the work, but also for the child to be able to pick up the subsidiary results worked out on the way.

The few technicalities that arise (e.g. percentages) should be treated as examples of general problems and not so much as types to be learnt.

The teacher who aims only at getting his pupils to do well in the term's examination, limited perhaps to the special work done, is not bearing his proper share of the burden; he must use

* On the other hand, to put up on the board a sample sum and to make them slavishly copy the style of that is quite contrary to the spirit we want.

the special work as a means of getting his pupils to tackle *any* and *every* question that may arise. Once more it is power that we must aim at developing. How many masters there are who are mere rule grinders! Their boys do well in the examinations at the end of term and the masters pat themselves on the back, and yet other masters who get the boys later find that they forget the rules and that they have little power of tackling anything new.

Finally I would lay stress on the fact that the ability to apply the fundamental processes in arithmetic depends ultimately on the power of understanding the wording of the question. Two things are necessary for this:

(i) the pupil must understand the technical terms that occur,

(ii) the pupil must be able to pick out from a mass of words the facts that are material.

As I have already pointed out, the meanings of any technical terms should be acquired by *vivâ voce* work.

The ability to solve "wordy" problems can only come ultimately by the child having plenty of practice with such questions; but this ability can be developed by class discussion.

One point I frequently notice is that a boy gets stuck in the middle of some question and gets some help from me; when he shows up his complete solution, he has practically slurred over the point about which I had to help him. In such cases I always say "The point over which you had to be helped is probably the most important point for you to bring out."

Words! Words! Words! In all applications and problems the wording is all important. If only a boy will get into writing fully, even too fully, it is easy enough to teach him where he might cut down his explanation.

One more point. Always insist on the boy's answer being written in the form of a sentence, e.g. "The eggs cost $2\frac{1}{2}d.$ each." Never allow "Answer $= 2\frac{1}{2}d.$"

MISCELLANEOUS PROBLEMS

Examiners, in endeavouring to test children's ability, have introduced new types of problems; teachers and text-book writers have then seized on them and taught children rules for solving questions of these types. Thus examiners have been driven to devise yet new types and the teachers and text-book writers have taught these. The vicious circle has gone on until many text-books have been overloaded with types of problems and rules for dealing with them.

The essential feature of a problem is that there should be something fresh and unfamiliar about it; if all problems are reduced to types and rules given for their solution, they cease to be problems. The teacher should approach problems in the spirit that they are problems and should remain problems; he should get his pupils to pick out the essential ideas in any problem they have to solve, and give them a little practice in similar problems, but the mixed collection of problems should be his great standby.

There are just a few types with which the class should be familiar. The idea of relative velocity should be given. I still remember spending most of a Sunday, as a boy, on a problem about a greyhound chasing a hare. Then there are questions about men doing work and taps filling baths; to many boys the solution of these is merely a trick, but the teacher should bring out the idea that to add or subtract two rates of doing anything we must first express the rates in terms of the same unit of time, so much per day or so much per hour. Apart from these two ideas I do not think there are any other special types that need be considered.

CHAPTER XI

PERCENTAGE

If a teacher asks a boy to explain what he means by 5 %, it is a common experience for the boy to say "£5 interest on £100." The lesson to be drawn from this is that we do not do enough work on percentage apart from money sums (interest in particular). I would advocate a frequent use of percentage before interest is taught at all.

In connection with practical work, boys should be taught that an error of an inch may be unimportant in measuring the length of a plank but is all important in measuring the length of a matchbox. It is useless to say "My result is 1 out, can I count it right?" It is essential to know the relative error, say 1 in 87. Even then it is hard to compare relative errors, e.g., which is the biggest relative error, 1 in 87, 0·6 in 48, or $\frac{3}{4}$ in 60?

Each error might be expressed as a decimal, but it is common practice to express them as errors per 100.

Th

$$\frac{1}{87} = \frac{1{\cdot}15}{100}, \quad \frac{0{\cdot}6}{48} = \frac{1{\cdot}25}{100}, \quad \frac{\frac{3}{4}}{60} = \frac{1{\cdot}25}{100},$$

and we speak of these errors as 1·15 per cent., 1·25 per cent., and 1·25 per cent. Later we may use % as an abbreviation. (I do not know the history of the symbol, but I always imagine that it is an abbreviated way of writing /100.)

The main thing is to lay stress on the fact that "x % of a thing means $\frac{x}{100}$ of it." Plenty of easy examples should be done to familiarise the idea of percentage and the use of letters should be encouraged.

Ex. 1. *Express $\frac{3}{8}$ as a percentage.*

$$\text{Let } \frac{3}{8} = \frac{x}{100},$$

$$x = 37\tfrac{1}{2},$$

is $37\frac{1}{2}$ %.

Ex. 2. 18 % *of the boys in a school wear spectacles; if there are* 250 *boys in the school, how many wear spectacles?*

18 % of the school is $\frac{18}{100}$ of 250.

∴ The number wearing spectacles is $\frac{18}{100} \times 250 = 45$.

Ex. 3. 18 % *of the boys in a school wear spectacles; if the number of boys wearing spectacles is* 72, *how many boys are there in the school?*

Suppose that there are x boys in the school.

Then $\frac{18}{100}$ of x wear spectacles.

But 72 wear spectacles,

$$\therefore \frac{18}{100} x = 72,$$

$$\therefore x = 400,$$

∴ there are 400 boys in the school.

Ex. 4. *An error of* 3 *in. is made in measuring a length of* 12 *ft., what is the percentage error?*

Suppose the error is x %.

$$\text{Then} \quad \frac{x}{100} = \frac{3 \text{ in.}}{12 \text{ ft.}} = \frac{3}{12 \times 12},$$

$$\therefore x = \frac{3 \times 100}{12 \times 12} \qquad 2 \cdot 1,$$

∴ the error is 2·1 %.

PROFIT AND LOSS

Profit and loss has loomed too large in arithmetic books and papers in the past. It should be introduced as an example of percentages and needs no elaborate treatment. The one essen-

tial point is that, unless otherwise stated, profit and loss are always estimated as percentages of the *cost price* and not of the selling price.

Ex. 5. *If a man makes a profit of 20 % by selling eggs at 4 for a shilling, what did they cost him?*

Suppose that each egg cost x pence.

Then 4 eggs cost $4x$ pence.

The profit was 20 % $= \frac{20}{100}$ of $4x$ pence $= \frac{4}{5}x$ pence,

$\therefore$ the selling price of 4 eggs was $(4x + \frac{4}{5}x)$ pence.

But the selling price of 4 eggs was 12 pence.

$$4x + \tfrac{4}{5}x = 12,$$

$$\therefore x = \tfrac{5}{2},$$

$\therefore$ the eggs cost $2\frac{1}{2}$ pence each.

CHAPTER XII

INTEREST, SIMPLE AND COMPOUND. STOCKS AND SHARES

I have not considered interest in the percentage chapter because I feel so strongly that the idea of percentage should be introduced and familiarised at least a term before interest is considered.

First of all I must remind teachers that we constantly talk of "interest at 5 %" when we mean "interest at 5 % per annum." I have often seen some members of a class confused because this had not been made clear.

Then I should like to say that I prefer to have nothing to do with a formula in teaching simple interest*. The idea should first be acquired from *vivâ voce* examples and then applied to examples involving easy calculation. After that, examples must be taken with heavier numbers; here there is much opportunity for choice of method: with easy fractions of a pound it may be best to use vulgar fractions, in other cases it may be wise to express shillings and pence as a decimal of a pound.

So long as a boy understands that 4 % of a thing is $\frac{4}{100}$ of it —and he should have already learnt this from other work on percentages—he should have no difficulty with the following examples.

* A public school master, who shares my views about the unintelligent use of formulae, recently told me the following. He asked a class of new boys how many of them had been taught to use the formula $I = \frac{PRT}{100}$; all but one had learnt it, so he said "Thank goodness one boy has been taught to use his wits instead of a mere formula. How were you taught to do inverse interest sums?" (The question to the one boy.) "Please, sir, I was taught $\frac{PTR}{100}$, please turn round." Roars of laughter from the class, in which the master joined heartily.

Ex. 1. *Find the simple interest on £250 for 5 years at 4 % per annum.*

S. interest on £250 for 1 year at 4 % per annum

$$= \tfrac{4}{100} \text{ of } £250.$$

$\therefore$ S. interest on £250 for 5 years at 4 % per annum

$$= £\tfrac{4}{100} \times 250 \times 5^*$$

Ex. 2. *A man finds that he has received £14 interest for half a year on a bank balance of £800, at what rate was interest allowed?*

(The question means "at what rate per annum was interest allowed." The words "per annum" ought to have appeared in the question.)

Let x % per annum be the rate.

S. interest on £800 for 1 year at x % per annum

$$= \frac{x}{100} \text{ of } £800.$$

$\therefore$ S. interest on £800 for $\frac{1}{2}$ year at x % per annum

$$= £\frac{x}{100} \times 800 \times \tfrac{1}{2}$$

$$= £4x.$$

But the interest is £14,

$$\therefore \; 4x = 14,$$

$$\therefore \; x = 3\tfrac{1}{2},$$

$\therefore$ the rate allowed was $3\frac{1}{2}$ % per annum.

Ex. 3. *Find the time, if the interest on £125 at 5 % per annum is £25.*

Let x years be the time.

S. interest on £125 for 1 year at 5 % per annum $= \frac{5}{100}$ of £125.

S. interest on £125 for x years at 5 % per annum

$$= £\tfrac{5}{100} \times 125 \times x.$$

But the interest is £25,

$$\therefore \; \tfrac{5}{100} \times 125x = 25,$$

* Do not simplify before this stage.

$$\therefore x = \frac{25 \times 100}{5 \times 125} = \ldots\ldots = 4,$$

$$\therefore \text{ the time is 4 years.}$$

Here is a harder example.

Ex. 4. *What principal will amount to £300 in 5 years at simple interest at 4 % per annum?*

Let £x be the principal.

S. interest on £x for 1 year at 4 % per annum

$$= \tfrac{4}{100} \text{ of } £x.$$

$\therefore$ S. interest on £x for 5 years at 4 % per annum

$$= £\frac{x}{5},$$

$$\therefore \text{ the amount is } £\left(x + \frac{x}{5}\right).$$

But the amount is £300,

$$\therefore x + \frac{x}{5} = 300,$$

$$\therefore x = 250,$$

$$\therefore \text{ the principal is } £250.$$

COMPOUND INTEREST

There is no difficulty about explaining the idea of compound interest.

It is frequently argued that compound interest is always worked from interest tables and that it is not worth teaching; but occasions do arise on which anyone may want to work out a compound interest sum when he has no tables, and the arrangment set out below (which I imagine most people use) is such a delightful instance of the use of contracted work that I consider it worth teaching.

Ex. 5. *To find the compound interest on £3675 for 3 years at 2½ %*.*

£	
3675	
73·50	2 %
18·375	½ %
3766·875	Amount at end of 1st year
75·33750	
18·83437	
3861·04687	,, 2nd ,,
77·22094	
19·30523	
3957·57304	,, 3rd ,,
3675·	Original sum
282·573	Compound interest for 3 years

STOCKS AND SHARES

I propose to say little about this subject except that it is a mistake to tackle it with young boys, even if they are bright. At the age of 15 or 16 it makes much more appeal than at 12, and many of the confusions that arise in the mind of the boy of 12 do not arise if the subject is first tackled at 15 or 16.

It is useful to get boys to ask themselves, when they see any sum of money mentioned in a question, "Is this money (to be invested), or interest, or stock?" If they will always ask themselves that question, and write after every sum of money they mention in their work one of the words "money," "stock," or "interest," the most general difficulty will soon disappear.

Just as with interest, and in fact all application of arithmetic, the first questions should be entirely *vivâ voce*; then should come questions with very simple manipulation (in the answers to which the wording is all important) and finally questions involving more manipulation.

* In this case it is easier to use the fact that $\frac{2\frac{1}{2}}{100} = \frac{1}{40}$, the interest for each year can then be found by dividing the amount by 40.

If the interest is only wanted "within a penny," it is enough to keep four places of decimals.

CHAPTER XIII

MANIPULATION IN MONEY AND OTHER SUMS

The methods of working all money sums should be made familiar by examples in which the manipulation is light. As soon as the method is familiar, examples may be taken involving more manipulation. It is comparatively simple to get boys to learn good methods for straightforward manipulation, e.g. multiplication and division of money, but it is extremely difficult to get them to realise the best method of tackling a more complicated piece of manipulation; indeed, much experience is necessary before it is possible to choose the best method in a particular case.

First of all I would lay down that, in general, *no manipulation should be done in the intermediate steps of an example.*

The arguments in favour of this course are:

(i) In very many cases it is quicker and easier to do all the manipulation at once.

(ii) It is easier at the final stage to choose the best method of attack, e.g. in a money sum to decide whether to decimalise the money, or to express the shillings and pence as a vulgar fraction of a pound.

(iii) If you decide to decimalise the money, it is easier to decide to how many places you must go when you have before you the numbers by which you have to multiply and divide.

(iv) In an examination the examiner will be able to see whether your method was sound (even if your answer is wrong) and is far more likely to give good marks for work set out clearly, even if there is a mistake in manipulation.

Secondly, it is seldom wise to reduce a sum of money to pence, unless only shillings and pence are involved.

Thirdly, in general, it is better to work in decimals of a pound unless the vulgar fractions involved would be simple.

To teach a class how to choose the best method, I advise special lessons on manipulation; these may arise from examples that have cropped up in other work, or the teacher may keep a collection of examples for the purpose. I should give an example to the class, let each boy do it as he pleased and then discuss the methods used.

Here is one example that I discussed with a class lately.

What is the cost of 1000 *shares costing* 17*s.* 10*d. each?*

Method a.

$$\begin{array}{rlr} 17s.\ 10d. \times 1000 & = 17\tfrac{5}{6} \times 1000s. & 167 \\ & = \tfrac{107}{6} \times 1000s. & \underline{107} \\ & = 107 \times 166{\cdot}6s. & 167 \\ & = 107 \times 167s. & \underline{1169} \\ & = 17869s. & 17869 \\ & = £893.\ 9s.\ 0d. & \end{array}$$

This violates many principles.

(i) The approximation was rash, it affected the third significant figure in a factor and therefore the third significant figure in the answer.

(ii) It is better to do multiplication before division, unless the division is exact.

(iii) It is seldom worth cancelling by dividing by an even number (other than 10), or by 5, if this involves losing a nought. It is so much easier to multiply by 100 than by 50 or 25 or even by 4.

Method b.

$$\begin{array}{rl} 17s.\ 10d. \times 1000 & = 17\tfrac{5}{6} \times 1000s. \\ & = \tfrac{107}{6} \times 1000s. \\ & = \tfrac{107000}{6}s. \\ & = 17833\tfrac{1}{3}s. \\ & = £891.\ 13s.\ 4d. \end{array}$$

Quite a good method in this case.

Method c.

$$17s.\ 10d. = 17\tfrac{5}{6}s.$$
$$= \tfrac{107}{6}s.$$
$$= \tfrac{107}{120}£.$$

$$\therefore\ 17s.\ 10d. \times 1000 = \tfrac{107}{120} \times 1000£.$$

Also quite good but slightly inferior to Method *b.*

Method d.

$$\begin{aligned} 17s.\ 10d. \times 1000 &= £0{\cdot}8916666 \times 1000 \\ &= £\ 891{\cdot}6666 \\ &= £\ 891.\ 13s\ 4d. \end{aligned} \qquad \begin{array}{r} 12\,\underline{)\ 10d.} \\ 20\,\underline{)\ 17{\cdot}8333s.} \\ £0{\cdot}8916666 \end{array}$$

Again quite a good method; probably the best if the class are happy with decimalisation of money.

Find the simple interest on £273. 12s. 9d. for 2 years at $3\frac{3}{4}$ % *per annum.*

S. int. on £273. 12*s*. 9*d*. for 1 year at $3\frac{3}{4}$ % per annum

$$= \frac{3\frac{3}{4}}{100} \text{ of } £273.\ 12s.\ 9d.$$

$\therefore$ S. int. on £273. 12*s*. 9*d*. for 2 years at $3\frac{3}{4}$ % per annum

$$= \frac{3\frac{3}{4} \times 2}{100} \text{ of } £273.\ 12s.\ 9d.$$

$$= \frac{15 \times 2}{4 \times 100} \text{ of } £273.\ 12s.\ 9d.$$

$$= \tfrac{3}{40} \text{ of } £273.\ 12s.\ 9d.$$

Probably the easiest method now is to multiply £273. 12*s*. 9*d*. by 3, and divide the result by 40.

However, if we decide to decimalise the money, it is clear that the result will be less than £273. 12*s*. 9*d*., so that it will be enough if we express that sum as a decimal correct to three places.

Here is an instance from dynamics of the advantage of postponing the manipulation.

To find the range of a projectile, boys found

$$\text{the time of travel to be } \frac{2 \times 2240 \times \frac{1}{\sqrt{2}}}{32} \text{ sec.,}$$

and the velocity to be $2240 \times \frac{1}{\sqrt{2}}$ ft./sec.;

one boy worked thus

$$\text{time} = \frac{\cancel{2} \times \cancel{2240}^{\cancel{560}\ 140} \times 0{\cdot}7071}{\cancel{32}_{\cancel{16}\ \cancel{4}}} = 98{\cdot}994 \text{ sec.,}$$

$$\therefore \text{ range} = 2240 \times 0{\cdot}7071 \times 98{\cdot}994 \text{ ft.;}$$

another worked thus

$$\text{range} = \frac{\cancel{2} \times \cancel{2240}^{\cancel{560}\ 70} \times \frac{1}{\cancel{\sqrt{2}}}}{\cancel{32}_{\cancel{8}}} \times 2240 \times \frac{1}{\cancel{\sqrt{2}}} \text{ ft.}$$

$$= 156800 \text{ ft.}$$

CHAPTER XIV

SIGNIFICANT FIGURES. DEGREE OF ACCURACY AND CONTRACTED METHODS

SIGNIFICANT FIGURES

There is a great deal of misunderstanding about significant figures among boys and also, I am afraid, among teachers of arithmetic who are not really mathematicians. It may be hard to give a good intelligible definition of significant figures but it is easy to make the idea clear by examples.

For a rough definition I would suggest the following:

All the figures in a number are significant figures except the noughts, which we put in, at the beginning or end of the number, to show the position of the decimal point.

Thus the distance of the sun from the earth is 94,000,000 miles.

Here there are two significant figures; the noughts are put in to show it is 94 *million* miles.

Again ·0057 has two significant figures, ·103 and ·00103 each have three significant figures, the nought between the 1 and the 3 is significant.

There is one difficulty however; 3000 may be a number of one significant figure or of four significant figures.

If I say to a man I will pay you £3000 for the house, I mean that I will pay precisely £3000 and not three thousand and some odd hundreds, etc., and in that case three thousand is a number of four significant figures, the noughts are significant.

On the other hand, if a man says to me "guess the value of that house" and I say £3000, I mean about £3000—and the three thousand is a number of one significant figure.

I am sometimes asked "Why introduce boys of thirteen to significant figures; why not always ask for a result to so many decimal places?"

The answer to that is: A length of 2·56 m. is the same as a length of 256 cm., and each is given to three significant figures. The number of significant figures in a length denotes the degree of accuracy of the length irrespective of the unit employed in measuring it.

DEGREE OF ACCURACY

It is important to consider to what degree of accuracy a result should be given.

Let us take an example.

In addition or subtraction of decimals

```
750·3
623·4
  3·685
 54·04
```

Here we are adding several numbers each of four significant figures. Note that in the column marked * we have a blank which would be filled up if the first number were given to a greater degree of accuracy. Therefore the result of adding that column is not trustworthy and should not be given in the answer.

The number of figures we can give in our answer does not depend on the number of significant figures in the given numbers, but depends entirely on which is the nearest column to the left with a gap in it.

In multiplication and division, however, the number of significant figures we can give in the answer depends on the number of significant figures in the numbers which we multiply or divide.

It often involves heavy work to decide how many significant figures we are justified in giving in a result. I would suggest working out several particular cases and so arriving at the following rough general rule.

If two or three numbers are multiplied or divided, count the number of significant figures in each, suppose that the smallest number of significant figures in any of the numbers is n, then $(n - 1)$ figures in the answer are fairly trustworthy, the n-th figure may be given but it must be realised that it is very doubtful.

I will first take a rough method of dealing with this.

Suppose we want to multiply 27·01 by 3·14:

$$\begin{array}{r}
27{\cdot}04\times \\
3{\cdot}14\times \\
\hline
81{\cdot}12\times \\
2{\cdot}704\times \\
1{\cdot}0816\times \\
\times\times\times\times\times\times \\
\hline
\end{array}$$

I have put in crosses to indicate gaps in which we should have figures if each number had been given to one more significant figure.

It should be obvious that we can trust no column in which a cross occurs, so that all we can say is that the answer is 85.

Again take 27·04 ÷ 3·14:

$$\begin{array}{r}
8{\cdot}61 \\
3{\cdot}14\times\,)\,\overline{27{\cdot}04\times} \\
25{\cdot}12\times \\
\hline
1{\cdot}92\cancel{0} \\
1{\cdot}884 \\
\hline
36\cancel{0} \\
314 \\
\hline
46
\end{array}$$

Here the cross has the same meaning as before, the $\not{0}$ means that in ordinary working we should insert a nought but that we do not know that the figure is a nought.

If we look at the 46 at the end, we note that the 4 comes from a column with a cross in it, so that we cannot get another figure in the quotient; indeed, in the line above, the 6 in 360 comes from a column with a cross, so that the 6 is untrustworthy.

Here we see that we are fairly safe in trusting two significant figures in the answer.

In both the above questions I have chosen numbers haphazard; sometimes teachers will find that they hit on numbers that seem to justify giving n figures (see the rule) in the answer and sometimes only $(n - 2)$.

Another rough method may be used on the blackboard by writing the last significant figures and all the figures that depend on them in red chalk (the danger colour). I will put out the above examples using italics instead of red:

$$
\begin{array}{r}
27{\cdot}0\mathit{4} \\
3{\cdot}1\mathit{4} \\
\hline
81{\cdot}1\mathit{2} \\
2{\cdot}70\mathit{4} \\
\mathit{1{\cdot}0816} \\
\hline
\end{array}
\qquad\qquad
\begin{array}{r}
8{\cdot}61 \\
3{\cdot}1\mathit{4}\,\big)\,27{\cdot}0\mathit{4} \\
25{\cdot}1\mathit{2} \\
\hline
1{\cdot}9\mathit{20} \\
1{\cdot}88\mathit{4} \\
\hline
\mathit{360} \\
31\mathit{4} \\
\hline
\end{array}
$$

Here we arrive at pretty much the same result.

Now for the really accurate method.

27·04 really means that the number lies				between	27·045
				and	27·035
3·14	,,	,,	,,	between	3·145
				and	3·135

Therefore the required product lies between

$$27{\cdot}045 \times 3{\cdot}145 \quad \text{and} \quad 27{\cdot}035 \times 3{\cdot}135$$

```
27·045                27·035
 3·145                 3·135
------                ------
81·135                81·105
 2·7045                2·7035
 1·08180                ·81105
  ·135225               ·135175
----------            ----------
85·056525             84·754725
```

From this we see that the product is 85 correct to two significant figures.

Again, in $27{\cdot}04 \div 3{\cdot}14$ the quotient lies between

$$\frac{27{\cdot}045}{3{\cdot}135} \text{ and } \frac{27{\cdot}035}{3{\cdot}145}\text{*}:$$

```
           8·62                       8·59
3·135)27·045              3·145)27·035
      25·080                    25·160
      ------                    ------
       1·9650                    1·8750
       1·8810                    1·5725
       ------                    ------
         8400                     30250
         6270                     28305
```

From this we see that the quotient is 8·6 to two significant figures.

The last method is laborious but exact. If a little drill is desirable in mere computation for the sake of speed and accuracy, it may be worth while to work out several such cases; half the class may work one case and the other half the other, and compare; but the question of the number of significant figures to be trusted is so difficult (unless we use the laborious method) that it is generally best to make boys realise the diffi-

* N.B. $\dfrac{\text{the largest value}}{\text{the smallest value}}$ and $\dfrac{\text{the smallest value}}{\text{the largest value}}$.

culty, and then to fall back on the rough rule I have given above.

Of course, many cases will occur in which boys give results to an absurd degree of accuracy.

E.g. more than four significant figures in an answer when the result depends on figures obtained from four-figure tables.

π is often taken as $\frac{22}{7}$ and a result given to even five or six significant figures.

In dynamics air resistance is neglected and g taken as 32 ft./sec.^{2} and results given to more than two significant figures.

In money sums results given to the nearest penny when four-figure tables have been used and the whole sum of money may be more than £100.

If an answer is required to three significant figures and works out to 28·35, it should be given as "28·3 or 28·4."

Sometimes a note should be made that the last significant figure of an answer is doubtful.

Examiners would often save examinees a lot of trouble if answers were asked for "within a penny" instead of "to the nearest penny"; to decide the *nearest* penny may involve working to many significant figures.

It is worth pointing out that with ordinary instruments it is hard to measure any length or weight to more than three significant figures. A chemical balance will weigh to a greater degree of accuracy; to measure to a greater degree of accuracy special precautions must be taken; in measuring with a steel tape we get no stretch, but a difference of 10° C. in temperature will affect the fourth significant figure. Again, suppose we are measuring the length of a room, are we to measure wall to wall or skirting board to skirting board? Even many practical men do not realise the limitations of their accuracy; a railway engineer once told me that in laying out a railway they worked with

absolute accuracy—I did not attempt to discuss the matter further with him.

It is also worth pointing out that in money sums we work to an extraordinary degree of accuracy: if we are dealing with a sum of say £5000 and give the result to the nearest penny, the result involves seven significant figures.

One other thing to point out is that the difference between two numbers each of four significant figures will probably not be correct to even three significant figures, e.g. $278{\cdot}4 - 276{\cdot}2 = 2{\cdot}2$, and certainly is not correct to more than two significant figures.

CONTRACTED METHODS

I was brought up on contracted methods and as a young master I taught them, but I am sure most of my labour as boy and master was wasted.

In multiplying together two numbers each of four significant figures a contracted method to my mind is no saving of time; to multiply it out in full and reject the end figures is quicker and more instructive.

In dealing with numbers of six or seven significant figures no doubt contraction is a saving of time, but there are few boys who will have to deal with real measurements involving numbers of more than four significant figures. The boy who ever has to deal with numbers of six or seven significant figures will probably have the brains to invent contractions for himself.

One delightful instance of the use of contraction occurs in compound interest, see p. 147.

PART IV

ALGEBRA

BY A. W. SIDDONS

The references are to Godfrey and Siddons' *Elementary Algebra*, Second Edition, 1920 (G. and S., *Algebra*). See also Siddons and Daltry's *Elementary Algebra*

CHAPTER I

WHY DO WE TEACH ALGEBRA?

Some people may hold that we teach algebra because of its use in other branches of mathematics; but, even if one of our aims is to acquire the use of a powerful weapon for other work, there are valuable ideas in algebra which have an educative value of their own, and the child must have some nearer goal to aim at than the hope of using it in other work after three or four years of drudgery.

Manipulative skill is of course essential for algebra itself as well as for its application to other subjects; but the acquisition of that is not the aim of the thoughtful far-sighted teacher*, and it certainly is not a suitable aim to set before the child. The child must have a nearer goal, and the manipulative skill it acquires, though to be used later, must be the skill obviously needed to reach the nearer goal.

What are the leading ideas to be acquired from elementary algebra? I would say (i) the solution of problems by equations, (ii) a power of generalisation and the use of formulae, and (iii) the idea of functionality.

SOLUTION OF PROBLEMS

I have placed the idea of solving problems first because it comes first historically. From the time of Ahmes (sometime between 1700 B.C. and 1100 B.C.) till comparatively modern times the solution of problems by equations has been the mainstay of elementary algebra. The use of an algebraical equation is a big step forward from purely arithmetical methods; it is a new idea to the child and fascinates him for a considerable time

* See pp. 43, 44.

GENERALISATION AND FORMULAE

Then there is the idea of generalisation, the generalisation of an infinity of particular statements into a single universal formula; to take a simple instance, twice seven is equal to seven times two, three times twelve is equal to twelve times three, and so on for ever. We can pack this endless multitude of truths into a sentence by saying that "the result of multiplication is independent of the order in which the two factors are taken." And we can compress the sentence into a formula, $ab = ba$. What could be neater?

There is no end to the series of general truths that find their simplest expression in an algebraical formula; every algebraical identity is shorthand for a sentence, and this sentence contains an infinity of particular statements. An algebraical formula looks unpoetical enough, but there is a certain aesthetic side to this power of pregnant expression that algebra possesses. It is analogous to Newton's grand generalisation of the movement of the heavenly bodies. Newton took as his problem the infinitely complicated movements of the sun and planets; movements that had been watched for thousands of years before people could disentangle the skein. At the end of his labours he said "In all these movements of earth, sun, moon and planets, I discern one thing happening, and one only. Every particle in the universe is attracting every other particle with a force varying inversely as the square of the distance." In this statement is contained the complete description of every movement. This was a grand generalisation of an infinity of particular phenomena. In a small way, there is an element of the same impressiveness in an algebraical formula.

The philosophical interest of an algebraical formula may not appeal to everyone; but there still remains the usefulness of a formula. No one who makes any practical use of mathematics in life can dispense with formulae. A formula is compressed in-

formation. To calculate the tax on a motor car, the horse-power needed to propel a ship, the range of a projectile under actual conditions, the strength of a girder, whatever the problem may be, there is a formula waiting to tell us about it. A formula economises thought. No doubt the use of formulae to economise thought is open to abuse in teaching; we do not always advise boys to economise thought. A wise teacher will have nothing to do with formulae in the earliest stages of a subject, e.g. in teaching mechanics. But the fact remains that in real life mathematical formulae are almost as necessary as the multiplication tables; and teaching that does not bring out this point is missing the whole *raison d'être* of algebra teaching.

FUNCTIONALITY

Another fundamental idea is that of functionality. For the sake of non-mathematical readers let us take the instance of the bicycle pump. Put your finger over the nozzle so that no air can escape; and then try to pump. You find that you can push the piston in, compressing the air inside. The farther you push the greater becomes the resistance. There is a mathematical relation between the force you exert and the distance you can push the piston; to push one inch you must exert a certain force; to push two inches you must exert a certain larger force (not necessarily twice as large) and so on. We say that the force is a function of the distance. The force and the distance are called variables, the one variable is a function of the other. Given the force, you can calculate the distance pushed; given the distance, you can calculate the force. This relation can be expressed as an equation connecting the two variables. Call one of them x and the other y and you have an equation connecting x and y. Furthermore, you can make all this visible to the eye. You can draw a curve, a graph as it is called, exhibiting to the eye the way in which the pressure increases with the distance. This graph is simply a visible form

of the equation, just as the written word is a visible form of the spoken word.

Whenever one measurable thing depends on another measurable thing, you have a case of functionality, you have an equation and you have a graph. To the mathematical eye, life is full of functions and graphs. With the idea of continuously changing quantity enters a fresh idea. Arithmetic does not present this idea, nor, in fact, did algebra as taught thirty years ago. But the idea should come into algebra *via* graphs and variation, it is bound up with the elementary idea of functionality.

THE ACQUISITION OF MANIPULATIVE SKILL

As I have said above, the acquisition of manipulative skill is not a suitable aim to set before the child; a manipulative skill that is not evidently useful to the child for some task which he has in hand is not a suitable stimulant. But the teacher with his longer vision will realise further uses for any particular pieces of manipulation, he must not fail to see that, whenever any piece of manipulation has arisen in a problem, the child understands it and acquires such a power to use it that it will be useful for the future as well as for the problem in hand. So, with the reservation that the child must see the use of any piece of manipulation, the teacher may have as an aim the acquisition of manipulative skill; but he must beware of teaching the child to manipulate just for the sake of manipulation.

After the School Certificate stage, those who require mathematics for its own sake or for science or engineering will need much drill at manipulation; but that is no reason for giving this drill before that stage to all pupils alike, whether they will ever need manipulative skill or not. It is mere waste of time to polish a tool for all pupils when the majority of them will never use it.

CHAPTER II

ELEMENTARY ALGEBRA OF THIRTY YEARS AGO

The English text-books of Algebra in vogue during the latter part of the nineteenth century have tended to degenerate into a mere farrago of rules and artifices, directed to the solution of examination puzzles of a stereotyped character having little visible relation to one another and still less bearing on practice.

From preface to Chrystal's *Introduction to Algebra.*

The typical start for algebra thirty years ago was the evaluation of a few expressions, that had no meaning for the pupil, followed by a long grind at the four rules (addition, etc.) and the removal and insertion of brackets. I was taken through this in the days of my youth and I saw neither rhyme nor reason in the subject until I reached problems.

Again, all through the usual text-books there were rules, many of them quite unnecessary, and the necessary ones were given far too soon. Frequently the text-books discussed a piece of manipulation and consolidated the discussion into a rule (given in prominent type) before the child met any examples involving that manipulation; the examples were then given to the class to be done by the rule. Text-book writers and teachers failed to realise that a rule should be the generalisation of the child's own experience; the necessity for the manipulation should have arisen (say in the solution of a problem), the teacher should then step in and discuss the manipulation and the child should learn how to do the manipulation in particular cases and should have some practice with it; from this practice the child might evolve a rule, but probably the statement of the rule would come days or weeks later.

The child was given little opportunity to invent rules for himself; he was given the rule, made to learn it like a parrot and then made to apply it to material that had little or no meaning for him. In fact, algebra was a sort of drill that had to be learnt; naturally it was not attractive to many children, and the rules were ill-digested and misapplied.

Manipulative skill seemed to be the aim of the teaching of thirty years ago, and the skill demanded went far beyond that which the non-specialist was ever likely to use.

CHAPTER III

MODERN METHODS OF STARTING ALGEBRA

The two modern methods of approach to the subject are (i) *via* the equation and the problem, (ii) *via* the formula.

Neither method can be pursued to the exclusion of the other. If the equation and the problem are made the early aim, a certain amount of symbolical expression, and so some formula work, must be introduced; and similarly, if the formula is made the early aim, equations will come in to some extent. In both methods the pupil has to acquire a certain amount of manipulative skill.

I prefer to make the equation and problem the *raison d'être* of the early work. First of all, in the history of the subject that came first, and history is generally a sound guide. Secondly, it is more attractive to the ordinary run of boys and girls—they have an aim and they enjoy it. Thirdly, in the hands of most teachers I consider it is the easier method; further, it is easy to bring in all the manipulation that is essential, and that manipulation grows gradually.

With pupils whose ability is above the average and with an able teacher, I can imagine that the formula method would give excellent results; but I feel that it puts manipulation into too prominent a position and brings in hardish manipulation earlier than the other method.

Nothing is really gained by acquiring manipulative skill unless it is *at the time* of value in use, i.e. in problems which the boy actually comes across. Judged by this test I feel that the equation-problem introduction to algebra is better than the formula.

Whichever method is used the wise teacher will see that, whatever manipulative process arises, the class masters it and applies it to enough abstract examples to use it not only for the problem in hand but also in future work. During the last twenty years one has heard that the pendulum has swung too far and that manipulative skill has been lost in the glorification of the problem; perhaps there is some truth in this, but it is not the fault of those who advocated the great change that has been made. The unwise teacher, who has grasped at the problem and forgotten everything except the problem, has neglected the opportunities that problems gave him for making his pupils skilful in manipulation.

CHAPTER IV

THE EARLY STAGES OF ALGEBRA

THE USE OF LETTERS

Before beginning algebra the boy should have been introduced to the use of letters to represent numbers. The introduction of a letter in *vivâ voce* work in arithmetic helps the boy to generalise, to think what are the essentials of the method he uses. Let us give some instances.

"How many pence are there in (i) 1*s*., (ii) 2*s*., (iii) 5*s*., (iv) x*s*.?

The answers to (i), (ii) and (iii) come almost without thinking. But (iv) will give a check.

Teacher. "What did you do in the case of 2*s*., 5*s*.?"

Pupil. "I multiplied the number of shillings by 12."

Teacher. "Then, what will you do with 153 shillings?
What will you do with x shillings?"

Similarly in all cases of changing from one unit to another (e.g. yards to feet, or yards to miles), the introduction of a letter helps to focus the pupil's mind on the essentials of the processes used.

Naturally letters will not be introduced when first tackling changes of units in arithmetic, but one day when the teacher has been doing some *vivâ voce* work the last questions might be such as I have suggested above.

Again with areas and volumes, e.g. What is the area of a rectangle 15 in. by 5 in., 70 ft. by 20 ft., l in. by 5 in., l in. by b in.?

In the case of fractions $\frac{3}{4} \times n$, $\begin{matrix}3\\4\end{matrix} \quad n$, $\frac{p}{q} \times 2$ all help to make the boy think of the essentials and to get the general rules.

It is very instructive to do a unitary method sum with letters. (See "Arithmetic," chap. IV, p. 101.)

If a *men can do a piece of work in* b *days, how long will it take* c *men?*

$$\begin{array}{lll} & a \text{ men can do it in } b \text{ days,} & \\ \therefore & 1 \text{ man} \quad ,, & b \times a \text{ days,} \\ & c \text{ men} & \dfrac{b \times a}{} \text{ days.} \end{array}$$

If we do this without the unit step, interesting points arise

$$\begin{array}{ll} a \text{ men can do it in } b \text{ days,} & \\ c \quad ,, \quad ,, & b \times \text{—}. \end{array}$$

Now suppose c is smaller than a, will the required number of days be larger or smaller than b?

Pupil. "Larger."

Teacher. "Then multiply by the larger and divide by the smaller, and the result is $b \times \frac{a}{c}$ days."

Pupil. "But how do you know a is larger than c?"

Teacher. "We supposed it so."

Pupil. "But suppose it is not?"

Teacher. "All right, we will work it out again, assuming a is smaller than c."

This leads to quite an interesting result.

PROBLEMS AND EQUATIONS

Algebra proper should, to my mind, begin with the solution of problems. As pointed out in chapters I and III, this is in accordance with the historical development of the subject; the aim in the early days of algebra was to find unknown (hidden) numbers, and it is interesting to note that an early symbol used, where we use x to-day, was a picture of a shut hand.

With the solution of problems as the goal for the present, the necessity for some manipulation crops up, and that is the proper time to give the manipulation required. Manipulation

that is not required is meaningless to the pupil, and so uninteresting; but the desire for manipulative skill is soon acquired, and then it becomes attractive to the pupil, and so profitable.

One of the greatest stumbling blocks in algebra with children who start with mere manipulation is to get them to solve problems; they get to regard algebra as consisting of a set of almost arbitrary rules, and it is really hard work to teach them to solve problems; whereas, if they start with problems, there is meaning in the manipulation, and they enjoy problems and solve them with success.

The first problems tackled may be so easy by arithmetic that they prefer not to use an x; but this preference can easily be overcome by explaining that these easy problems are necessary just to get the ideas; the promise of power to solve harder problems is enough to spur on the class.

I remember on one occasion tackling the following with a class of beginners:

"*Two motor cars can run one at* 15 *miles an hour the other at* 18 *miles an hour. If the faster car sets out to catch the slower when the latter has* 9 *miles start, when will it catch it up?*" (See G. and S., *Algebra*, Ex. 1, i, Ques. 10.)

They did not see how to tackle it by arithmetic. I did not let them try too long, or they might have done it. After solving it by algebra, I showed them that it could be done by arithmetic, the one car had 9 miles to catch up and it caught up 3 miles every hour; they were delighted with both solutions.

In this early stage it is very essential to insist on careful wording* in the solution of problems, and of course the solution of equations must always be based on the fundamental axioms. Never let them take things across from one side of an equation to another. Always lay stress on doing the same to each side.

* This is valuable not only for the sake of the problem in hand but also for its indirect effect on the solution of problems in arithmetic.

E.g. $x + 6 = 9$.

Teacher. "What is the nuisance?"

Pupil. "The 6 on the left."

Teacher. "What do we do?"

Pupil. "Subtract 6 from each side."

Good teaching at this stage will show up when the boy gets on to literal equations (see chap. XII).

For some time it is wise to discuss problems *vivâ voce* before the class is set to solve them.

Take the problem on p. 171.

Teacher. "What shall we take x to be?"

Pupil. "The time" or "When it catches up."

Teacher. "But x must stand for a number."

Pupil. "The number of hours before the faster car catches up the slower."

Teacher. "How far does the faster car go in 1 hour? In 2 hours? In x hours?"

"How far does the slower car go in x hours?"

"If the slower car had 9 miles start, how far will it be from the starting place after x hours?"

Such a preliminary discussion will help the boy to put out his argument nicely. It is often useful to discuss several problems in this way and then set them for out of school work.

Whenever a piece of manipulation has arisen out of a problem, there should be some drill on the manipulation: the class are ready for it and will see its purpose and should enjoy it. Then they are equipped and ready for equations and problems involving similar pieces of manipulation.

The keen teacher who is doing this work for the first time, even if he has had experience in teaching the later work, will be surprised and interested to find how many little points arise which need explanation, and which he would have assumed to be obvious.

E.g. It is necessary to explain that $5x$ is a conventional form for 5 times x, or $5 \times x$, or $x \times 5$.

It would be a mistake to make a list of these and point them out beforehand; the real teacher will deal with them as they arise. But it is a very sound thing for the teacher to make a list of the difficulties that are likely to arise, and of the various pieces of manipulation that have got to be mastered, and to tick off each when it has been dealt with and mastered by the class.

Here is a possible list for the points that arise before negative number is introduced: $2x = 2 \times x$, $27 = 2 \times 10 + 7$, generally the number whose digits are t and u is $10t + u$; $19x + 13x = (19 + 13)\,x = 32x$; $8y \times 4$, $4 \times 8y$, $5\,(x \pm y)$, multiplication by 0.

In the case of most of the difficulties that arise, the teacher will merely go back to the concrete, or put numbers for letters, $5x + 3x$ can be simplified at once if the teacher merely says "5 donkeys + 3 donkeys equals how many donkeys?", or more formally,

5 lengths of 12 inches each and 3 lengths of 12 inches each give how many lengths of 12 inches? 5 lengths of x inches each and 3 lengths of x inches each give us....

I cannot lay too much stress on the importance of not hurrying the boy over this stage of the work. It is useless to grind him *ad nauseam* at a piece of manipulation that he does not understand; if he cannot tackle the problems which he is set, then he must have easier problems.

NEGATIVE NUMBERS

There has been some controversy as to when negative numbers should be introduced. I do not think the time of its introduction is a vital matter; if the desire for it arises early, it is a mistake not to seize the opportunity. Some years ago with a

class of beginners a problem about distances up and down hill gave us the equation

$$3x + 6 + x - 20 = x + 7 + x - 3,$$

they took $3x + x$ as $4x$, the $x + x$ as $2x$ and the $7 - 3$ as 4 and most of them solved correctly dealing with the 6 and -20 separately. One boy, however, asked why he could not deal with the $6 - 20$ as he had done with the $7 - 3$. Here was the opportunity to introduce negative numbers.

Teacher. "6 yards uphill and then 20 yards downhill is the same as what?"

That started us off and we discussed a good deal of the negative number chapter; after that we went back to the earlier chapter and found that we had much more power with equations.

I do not propose to enlarge here on negative number, most modern text-books treat it adequately and the teacher can follow them and introduce his own illustrations as well. The main thing he will work on will be the scale

$$-3 \qquad\qquad -1 \quad 0 \quad 1 \qquad\qquad 3$$

He can disguise it and make it more attractive by making the zero the milestone nearest to the school and 1, 2, 3 the numbers of the milestones towards London, -1, -2, -3 the numbers of the milestones in the opposite direction, or he can introduce temperatures and lifts. If his illustrations come out of his head instead of out of the book, so much the better.

I remember, many years ago, in the early days of motors, walking with a non-mathematical colleague who was teaching the beginnings of algebra; he had just asked me for a really good illustration that $-(-x) = +x$ when we saw a car turn round in a wide road, get into difficulties with its gears and start moving backwards. That gave me the idea which will be found in G. and S., *Algebra*, § 30.

The ideas of negative numbers must not be hurried; though I have discussed them here in a few lines, weeks of quiet work are needed to get the class clear and efficient in their use.

It is not essential for the class to be able to reproduce explanations; what is essential is that they should feel the reasonableness and self-consistency of the rules arrived at; for, after all, $(-2) \times (-3) = +6$ depends on a rule. The philosophy of the subject is best dealt with at a later stage, and then only with the abler boys.

FRACTIONS WITH NUMERICAL DENOMINATORS

A problem may now be taken that involves fractions with numerical denominators. Such fractions present no difficulties if vulgar fractions have been properly taught in arithmetic.

The *vivâ voce* work, e.g. G. and S. *Algebra*, Exs. $3a$, $3b$, and $3c$, is merely a slight extension and consolidation of their arithmetical knowledge.

In written work the one pitfall is in dealing with a fraction such as $-\frac{y-2}{2}$ and this is easily overcome if brackets are used.

$$\text{E.g.}\quad \frac{2y-3}{3} - \frac{y-2}{2} = \frac{2(2y-3)}{6} - \frac{3(y-2)}{6}$$
$$= \frac{2(2y-3) - 3(y-2)}{6}.$$

Sooner or later brackets will be dropped and the following mistake is bound to occur:

$$\frac{2y-3}{3} - \frac{y-2}{2} = \frac{4y-6}{6} - \frac{3y-6}{6}$$
$$= \frac{4y-6-3y-6}{6}.$$

All that is necessary is to point out that $3y - 6$ is as good as

in a bracket (the line underneath is almost like a vinculum, $\overline{3y - 6}$). They understand at once, but the mistake must be watched for at present.

A particularly vicious double mistake occurs so frequently that I think it must be persistently taught in some preparatory schools,

$$\frac{x}{3} - \frac{x-4}{4} = \frac{4x}{12} - \frac{3(x-4)}{12} = \frac{4x}{12} - \frac{3x+12}{12} = \frac{4x - 3x + 12}{12}.$$

The answer is correct, but the method is utterly pernicious.

Two other mistakes must be noted and watched for.

(i) After solving fractional equations, boys will lose their denominators when simplifying fractions, e.g.

$$\frac{3x-1}{5} - \frac{2x-3}{3} = 3(3x-1) - 5(2x-3)$$

is of quite common occurrence.

(ii) In fractional equations, e.g

$$\frac{3x-1}{5} - \frac{2x-3}{3} = 1,$$

never pass

$$\frac{3(3x-1) - 5(2x}{15} \qquad = 15$$

It is meaningless; you cannot divide an equation by 15.

It is amazing to me to find how many boys have quite unnecessary rules for solving fractional equations.

In a *vivâ voce* examination lately I asked boys to say how they would solve

$$\frac{2x-1}{6} - \frac{x+4}{9} = 5.$$

Almost invariably they said "Find the L.C.M." but they did not say of what, but let that pass. Many of them said "6 goes into the 18 three times so you multiply the first fraction (*sic*) by 3, and 9 goes into 18 twice so multiply the second fraction

by 2 and multiply the 5 by 18"; no doubt they would have done it correctly, but further questioning showed that very few realised that they had multiplied both sides of the equation by 18. Several boys flatly denied that they had done so and seemed to think that I was a stupid fool for suggesting that they had.

I would suggest to teachers that they should make boys for some time write out the solution of such an equation as follows and never mention L.C.M:

$$\frac{2x-1}{6} - \frac{x+4}{9} = 5.$$

Multiply both sides by 18,

$$\therefore\ 18 \times \frac{2x-1}{6} - 18 \times \frac{x+4}{9} = 5 \times 18^{*},$$

$$\therefore\ 3(2x-1) - 2(x+4) = 90,$$

$$\therefore\ 6x - 3 - 2x - 8 = 90,$$

TESTING ROOTS OF EQUATIONS

As soon as boys solve equations they should be taught to check their solutions and to put the work out intelligently. They must take each side of the equation separately and find its value for the value of x found.

Thus $$3x + 6 + x - 20 = x + 7 + x - 3,$$

$$\therefore\ x = 9.$$

When $x = 9$, L.H.S. $= 27 + 6 + 9 - 20 = 22$,

R.H.S. $= 9 + 7 + 9 - 3 = 22$.

* Definite practice should be given with examples such as

Simplify $$12\left(\frac{x+7}{4} - \frac{x-7}{3} + \frac{x}{6} - 3\right).$$

Never allow them to work with both sides at once and ultimately get down to $0 = 0$.

It seems to me important to teach them to use methods in their check that are different from those of their solution, e.g.

$$2(x-1) - 3(2-x) + 4(1-x) = 0,$$

$$\therefore\ x = 4.$$

The check should *not* be:

When $x = 4$,

L.H.S. $= 2(4-1) - 3(2-4) + 4(1-4)$

$$= 8 - 2 - 6 + 12 + 4 - 16 = 0.$$

If the boy has made a mistake in removing his brackets in solving the equation, he is likely to make the same mistake in his check so that he is led to think his solution correct.

His check should be:

When $x = 4$,

$$\text{L.H.S.} = 2(4-1) - 3(2-4) + 4(1-4)$$
$$2 \times 3 - 3 \times (-2) + 4 \times (-3) = 6 + 6 - 12 = 0.$$

Let us take a fractional equation,

$$x + \frac{3x-9}{5} = 11 - \frac{5x-12}{3},$$

$$\therefore\ x = 5\tfrac{1}{7}.$$

Check. When $x = 5\frac{1}{7}$,

$$\text{L.H.S.} = 5\tfrac{1}{7} + \frac{15\frac{3}{7} - 9}{5} = 5\tfrac{1}{7} + \frac{6\frac{3}{7}}{5} = 5\tfrac{1}{7} + \frac{45}{5 \times 7} = 5\tfrac{1}{7} + 1\tfrac{2}{7} = 6\tfrac{3}{7},$$

$$\text{R.H.S.} = 11 - \frac{25\frac{5}{7} - 12}{3} = 11 - \frac{13\frac{5}{7}}{3} = 11 - \frac{96}{3 \times 7}$$
$$= 11 - \tfrac{32}{7} = 11 - 4\tfrac{4}{7} = 6\tfrac{3}{7}.$$

Here the work of the check is quite unlike that of the solution, so that there is little danger of having made the same type of mistake in the solution and in the check.

Another thing to watch for is that often boys will check an intermediate step in their work, e.g. in a fractional equation, they will check the equation which they get after fractions have been removed All that their check shows is that there is no mistake in the last few lines of their work, a mistake before the line checked would not be revealed. They must always **check the equation as given in the question.**

SIMULTANEOUS EQUATIONS

Simultaneous equations will naturally be introduced by means of a problem or problems.

It is most important to bring home to the boy what is meant by solving a pair of simultaneous equations. Suppose we want to solve

$$2x + y = 10 \qquad \text{......(i)},$$

$$x + 2y = 8 \qquad \text{......(ii)}.$$

First consider (i). We can write down hundreds of pairs of values of x and y that satisfy this equation.

Again we can write down hundreds of pairs of values of x and y that satisfy (ii).

Our puzzle is to find the one pair that satisfy (i) and also at the same time (simultaneously) satisfy (ii) We could do this by searching through the two lists of pairs of values; but there are methods that are quicker and more certain.

All this is best done first with a definite concrete problem. See G. and S., *Algebra*, § 43.

The method of substitution is worth a lesson and a few equations may be solved in that way. I know the method will be forgotten almost at once, but a useful seed will have been sown for the time when simultaneous quadratics are tackled later.

Example

$$40x + 20y = 320 \qquad \ldots\ldots\text{(i)},$$

$$x + y = 10 \qquad \ldots\ldots\text{(ii)},$$

(ii) $$y = 10 - x,$$

$\therefore$ (i) $$40x + 20\,(10 - x) = 320, \text{ etc.}$$

In the ordinary method of solving simultaneous equations, it is a strange fact that boys are either taught, or have a natural preference for subtracting.

E.g. It is not uncommon to find the following:

$$2x - y = 1 \qquad \ldots\ldots\text{(i)},$$

$$x + 3y = 11 \qquad \ldots\ldots\text{(ii)},$$

(i) $$-6x + 3y = -3,$$

therefore subtractin $$7x = 14.$$

Instead of $$2x - y = 1,$$

$$x + 3y = 11,$$

(i) $$6x - 3y = 3,$$

therefore adding $$7x = 14.$$

To avoid this, it is a good plan to start with several cases in which addition is simpler than subtraction.

It is also wise to run through a whole set of examples such as G. and S., *Algebra*, Ex. 4*d*, and discuss whether it is easier to eliminate the x or the y; this discussion should be done *before* the class starts to solve any of the equations.

All the work referred to in the last few paragraphs (fractions with numerical denominators, simultaneous equations) paves the way for the solution of more and more difficult problems. It is perhaps surprising what a stimulus the solution of problems can be to boys who are taught sensibly, and the obvious thing is to use that stimulus.

The class will settle down to problems of various types, and the teacher's task will be mainly to help the individual over his difficulties and mistakes.

In the course of the work referred to in this chapter the teacher must have introduced a certain number of technical words; he should see that they are always used correctly but they should not be the subject of formal definition; the boy should be learning to use the language of algebra just as a child learns the use of his mother tongue and just as he should learn to use the language of geometry (see "Geometry," p. 255; also chap. VII, p. 190).

TIME FOR THE WORK OF THIS CHAPTER

The early stages of any new subject are all important and it is the greatest mistake to hurry a boy through them; his whole attitude towards the subject for life will be affected by the start he gets in it.

In geometry a boy starts by dealing with things with which he has already got some acquaintance, but algebra seems to be a much more abstract subject so that it is specially important not to force the pace in the first few terms at algebra.

From my own experience with a very backward class starting the subject at fourteen, and from what I have observed in preparatory schools starting boys at eleven, I consider that in a first term a class may be expected to cover the work I have referred to on pp. 170–174, i.e. they will have dealt with problems and equations and the manipulation that arises therefrom, and will have made some start at the idea and use of negative numbers.

In their second term they should have covered the work of the rest of this chapter, i.e. fractions with numerical denominators and simultaneous equations.

In their third term most of them will need to do a good deal more work over the same ground, but this can be brightened by introducing index notation and perhaps starting graphs.

The young teacher must be specially warned against going too fast; the work throughout may be described as problems

with the manipulation that arises from the problems, but each piece of manipulation must be mastered as it arises and the teacher must be careful not to set problems that provoke new pieces of manipulation before the old is mastered. I remember some years ago a brilliant young historian being appointed to take a middle form at a public school; at the end of three weeks he appealed to a colleague about the history for his form, he had finished the period allotted for the whole term, what should he do next? The colleague's reply was "Examine them on it and then do it again more slowly, but next term do not go so fast." The examination proved the wisdom of the colleague's advice. I can quite imagine the young mathematical teacher getting through the work referred to on pp. 170–173 and feeling ready to start on negative numbers after three weeks or half a term, but he would find that much of the elementary manipulation he should have done was unsound and, sad to say, he has taken all the jam of the work of the term and the class will feel bored and think they are merely revising. The wise teacher will have consolidated the manipulation as it arose and will still have some of the jam left for the rest of the term.

It is really sad to find boys of fourteen who have been learning algebra for two or three years who are unsound on work they should have done in their first term at the subject; naturally they are bored with covering the old ground again.

Do not hurry over the early stages, and consolidate the manipulation as it arises.

CHAPTER V

PROBLEMS

Boys who have been taught the beginnings of algebra in the way I have suggested, i.e. by solving easy problems from the start, seem to go on quite naturally to the solution of harder and harder problems as they occur; but boys who started algebra with blind manipulation and without understanding what the subject is driving at seem to have extraordinary difficulty with problems when they come to them.

A colleague who has had great experience and success in dealing with stupid boys, often taught on bad lines until they have got into his clutches, has written the following account of a typical lesson with such boys. With most boys such a detailed lesson should be unnecessary, but there are ideas to be got from this lesson that may be useful at times even with good boys.

"I buy 4 apples at $2d.$ each. What new fact can you give me?"

"A man goes for a walk at the rate of 3 miles per hour and continues for 2 hours. What question am I going to ask you?"

The correct reply will usually be forthcoming and questions of this sort seem to help those boys who find great difficulty in solving algebraical problems. A few questions similar to the above will give the boy ideas as to how to combine facts; and the main trouble in problems is to find out how the given facts can be combined.

The following may serve as an example:

"*A man buys a case of oranges at* 8*d. per dozen. He finds* 54 *spoilt, and selling the rest at* 7 *for* 5*d.*, *he loses* 2*s.* 6*d. on the whole. How many did he buy?*" (G. and S., *Algebra*, Ex. 3*f*, No. 7.)

Start by writing down

"Let x be the number of"

and then consider what it is of which we are asked to find the

number. In this question we have to find the number of oranges he bought; so complete the line, which now reads

"Let x be the number of oranges he bought."

Now pick out the separate facts as given in the question and write them down as you find them.

"1. Oranges cost 8*d*. per dozen."

"2. 54 are spoiled."

"3. Oranges sell at 7 for 5*d*."

"4. He loses 2*s*. 6*d*. on the whole."

How can we combine our top line which describes x with any of the four facts?

If a man buys 60 oranges at 8*d*. per dozen, what new statement can we get from a combination of these facts?

We can say that he paid $\frac{60}{12} \times 8$ pence for them.

In the same way we can use fact 1 to write down the cost price of the x oranges. So we now have a new fact.

"5. The oranges cost $\frac{8x}{12}$ pence."

We have finished with fact 1 so cross it out.

How about fact 2?

Suppose we have bought 60 oranges, how can we combine that with the fact that 54 are spoiled? We can now say how many were good and therefore saleable. So,

"6. $(x - 54)$ oranges are sold."

Cross out fact 2 it is finished with.

Fact 3 tells us that 7 are sold for 5*d*. so we can now combine 3 and 6 and have

"7 $5\frac{x - 54}{7}$ pence is the price for which the good oranges are sold."

Our remaining work now appears thus:

"Let x be the number of oranges bought."

"1. (Crossed out) "

"2. (Crossed out)."

"3. (Crossed out)."

"4. He loses 2*s*. 6*d*. on the whole."

"5. The oranges cost $\frac{8x}{12}$ pence."

"6. (Crossed out)."

"7. $\frac{5}{7}(x-54)$ pence is selling price."

Of our original facts, as given us in the question, only one remains.

Copy this down omitting the numerical part (2*s*. 6*d*.).

"8. He loses on the whole."

Fill in the blank by using facts 5 and 7. Now we can find what money is lost by subtracting the selling price from the cost price and can fill in the blank.

"8. He loses $\frac{8x}{12}-\frac{5}{7}(x-54)$ pence on the whole."

But the wording of 4 and 8 is identical, so the sums of money must be the same. Therefore $\frac{8x}{12}-\frac{5}{7}(x-54)$ pence is the same as 2*s*. 6*d*.

Here then is our equation:

$$\frac{8x}{12}-\frac{5}{7}(x-54)=30.$$

The above method makes an easy example become very easy and with practice in the preliminary questions success may soon come to the boy. When he has got two or three of these easy questions right he will have gained confidence; and, if the backward boy can once feel that he has some method by which he can tackle a question, he will usually make an attempt.

At first the questions should be chosen so that x can be taken as the number which must be found for the answer. The whole idea should be to let the boy feel that he has got some definite starting point, and that the business in front of him is merely

to use his common sense in combining the available facts in pairs until he has only one of his original facts left; and then he must combine in some way what remain of his new facts to replace or correspond with that one remaining fact.

This may seem mechanical, but it helps the boy to get the easy questions right and so to obtain confidence in himself; and when he has once acquired confidence, there is some hope of his making a more or less creditable attempt at more involved questions.

This method gives him the beginning of the idea of combining two facts. Perhaps one of the results may be that, instead of showing up a collection of apparently unconnected addition, subtraction, multiplication and division sums, from a combination of which (with some aid from the end of the book) the correct answer may appear, he will find himself putting down some explanation of his work and even getting the correct answer honestly.

One useful thought to some boys is that to get an equation we have to express some one thing in two different ways and equate the two expressions.

CHAPTER VI

MISCELLANEOUS TEACHING POINTS

ATTENTION AND THE USE OF SCRAP PAPER

Much of the early work in algebra, as in any new subject, must be done *vivâ voce*; but, when a class is merely answering questions or watching the master work on the board, they do not get the best out of the work: their attention is passive not active.

Whenever *vivâ voce* work is being done in algebra, I would advise that each boy should have a piece of scrap paper before him and that, though some questions should be addressed to individual boys, others should be addressed to the whole class and each boy should write down the answer. This is especially helpful in all the early work in algebra.

Suppose the master is solving an equation on the blackboard, different boys do the different steps; at various stages each boy should write down the next step on his scrap paper. Not only does this procedure keep better attention, active attention instead of passive, but, if a boy has made a mistake or does not follow a step, he is much more likely to ask for an explanation than he would be if the master alone wrote on the board to the dictation of two or three members of the class.

The vital step of a piece of manipulation should often be done in this way by the whole class, e.g. in simplifying

$$\frac{2x-1}{2}-\frac{3x-2}{3}=\frac{6x-3-6x+4}{6},$$

each member of the class should write down his result

Teacher. "You all have 6 underneath, I suppose. Jones, read off your numerator." If he reads off correctly.

Teacher. "Hands up those who disagree." Someone has got -4 instead of $+4$. "Hands up those who have got -4."

The point can be dealt with at once, and the effect produced is much greater than would be produced by handing back the boy's paper on the next day with the mistake carefully marked in blue pencil or red ink.

I do not mean to imply that the class will not have to do a lot of grind in the ordinary way, drill is essential and plenty of it; but, in the early stages of learning to do a piece of manipulation, the scrap paper method which I have advocated is quicker and it ensures better attention.

The scrap paper should not be allowed to be untidy, though in general the master would not look at it or attempt to mark it.

CHAPTER VII

INDEX NOTATION AND TERMINOLOGY

INDEX NOTATION

So far our aim has been the problem, and in pursuing that we have learnt to solve a variety of equations, we have studied negative number and have acquired a good deal of manipulative skill. But we have been limited to using a single letter: we have not considered such expressions as $10x^2y$; to the child such expressions were not likely to appeal, and to manipulate them would have been meaningless, but now we have to extend our field.

The necessity for some such expressions can be made obvious by considering the area of a square of side x inches or the volume of a cube; and it is a natural step forward to more complicated expressions.

The object of this chapter is to introduce such expressions and to learn to manipulate them and evaluate them. The notation has already been used in arithmetic.

As to their evaluation we have only to go back to the fundamental idea, e.g. $11x^3y^2$ is merely shorthand for

$$11 \times x \times x \times x \times y \times y,$$

and it is easy to find the value of this when values of x and y are given.

The two points of manipulation to be learnt at this stage are

(i) The index law (in general terms $10^p \times 10^q = 10^{p+q}$, though it would hardly be put in this general form at this stage).

(ii) The child must learn that like terms can be added and subtracted, but that $5x^2 + 7x^3$ is incapable of simplification at this stage.

It is a mistake to have one large dose of this work and to imagine that the battle is then won. A better course is to have

a few lessons at it and then an occasional lesson; it affords excellent material for "the nine questions*" and the results there will show how much grind is needed.

TERMINOLOGY

It is convenient at this stage to reconsider some technical terms that should have been gradually introduced into the previous work, and to introduce a few new ones.

It would be a mistake to introduce all these terms at one time: they should slide into the child's vocabulary naturally, but the time has come to collect them together and see that they are clear.

* See p. 59.

CHAPTER VIII

GRAPHICAL WORK

Graphical work is full of interest and very fruitful in developing and consolidating ideas, but I doubt whether its full interest or point can be obtained by a teacher who has not done a good deal of graphical work and who has no knowledge of the calculus.

When graphs were first being boomed (soon after 1900) many teachers, and books too, began to teach a sort of watered down co-ordinate geometry. On the one hand, the work aimed at a technical skill which would never be used by the majority of the pupils; on the other, pairs of simultaneous linear equations were solved graphically by the dozen, and the poor pupil thought it foolish because he knew a better method. The work seemed to lead up to analytical geometry instead of the calculus.

The true aim of graphical work should be very different. It should give the idea of the interdependence of two related quantities; it should give ideas of continuity; it should illustrate the self-consistency of mathematics; it should give a power of dealing with questions that are impossible or too difficult by ordinary algebraical methods. Graphs also give the pupil an excellent opportunity of criticising his own work: a graph that suddenly moves off in an unexpected direction or a point that spoils the smoothness of a curve soon leads the pupil to suspect a mistake.

Let me quote from a Board of Education circular (No. 884, "The Place and Use of Graphs in Mathematical Teaching"):

"The primary use of a graph is to exhibit to the eye a series of simultaneous values of two quantities. Thus, if we know the temperature at each hour of a day, we can, in some respects, seize the facts more readily from a plotted, than from a merely tabular record.

"But, if instead of the separate individual observations, we have a continuous record (the barograph is an example now familiar to most people), the record shows more than any table could; while the more complete we make the table, the more difficult does it become to appreciate its significance owing to the mass of figures involved. Further, the continuous record or graph at once suggests much more than records of the actual temperature at various times, for it forces on the attention the manner of change. This can indeed be derived from the table, but only with difficulty, but from the graph there may be read off at once such a statement as the following: 'the temperature was lowest about 4.30 a.m., it increased slowly till about 8.0, then more and more rapidly till about 12.0, more slowly again till 2.0, remained practically stationary till 3.30, when it began to decline, at first slowly, then more rapidly,' and so on.

"Again, if we plot the amount of a sum of money at a fixed rate per cent., first at simple interest, then at compound, anyone will appreciate the statement 'in the one case the amount increases uniformly, in the other more and more rapidly as time goes on.'

"It will be found that pupils have little or no difficulty in connecting greater or less rate of change with greater or less steepness of the graph, and this without any attempt to specify the rate numerically. Such attempt would indeed fog and spoil their intuitive comprehension of varying rapidity of change."

THE FIRST STAGE

The graphing of statistics must comē first. This has really nothing to do with algebra, but there are many lessons to be learnt from that work.

The importance of stating what the graph represents.

The necessity for graduating the axes and indicating the units employed.

The choice of appropriate scales.

Many lessons in decimals.

The distinction between a graph consisting of isolated points (e.g. a graph showing rainfall in different years) and a continuous graph in which intermediate points have a meaning.

Limits of accuracy.

Errors of observation.

Various deductions that can be made from the finished graphs.

As this stage bears no real relation to algebra, it may be taken at any time that is convenient; for the sake of spreading out the interest aroused by graphical work, I would suggest that the first and second stages should not be done in the same term.

Graphical work takes a lot of time, and it is useful to preserve the graphs drawn for future reference so that it is advisable to work in note books ruled for the purpose, or better still in Durell and Siddons' *Graph Book**.

THE SECOND STAGE

The next stage, the graphing of a function, is truly connected with algebra.

To graph innumerable straight lines is dull; it is much better to start by graphing a curve and the curve should arise out of a problem. See G. and S., *Algebra*, § 63, or Durell and Siddons, *Graph Book*, p. 32.

In the course of such a problem, besides getting the graph, we are developing ideas of a function, of the relation between a function and its graph and the idea of continuity of a graph.

After a full discussion of such a graph and the various deductions that can be made from it, the class should be eager to graph other functions not necessarily connected with particular problems.

Two points may be noted:

(i) It is often more convenient to take a function in the form of factors, thus $(x - 2)(x - 4)$, rather than in the form

* Published by Messrs Geo. Bell and Sons.

$x^2 - 6x + 8$. The factor form brings out different points, e.g. it shows how reasonable it is that

$$1 \times (-1) = -1 \quad \text{and} \quad (-1) \times (-1) = +1, \text{ etc.}$$

(ii) It is worth taking some trouble about making the table for a graph. Avoid drawing too many horizontal lines in the

x						
$x-2$ $x-4$						
$(x-2)(x-4)$						

table. In the table above there is less likelihood of making mistakes than there would be with an extra line drawn between the $(x - 2)$ line and the $(x - 4)$.

Again, in the table for the graph $3x^3 - 2x^2 + 11x$ the x^2 and x^3 lines are useful and, in the final addition to get the values of

x				
x^2 x^3				
$3x^3$ $-2x^2$ $11x$				
$3x^3 - 2x^2 + 11x$				

the function, mistakes are less likely than they would be if more lines were ruled across.

To get the full value and interest out of graphing functions, many questions (preferably *vivâ voce*) should be asked about the graphs.

What are the values of the function for given values of x?

For what values of x has the function a given value?

For what value of x is the function greatest (or least)?

As x increases, where does the function increase and where decrease?

Where is the function increasing most rapidly?

Etc., etc.

After graphing some curves it is interesting to draw the graph of $2x + 1$, say, and see that it gives a straight line. By noticing that if we take any value for x and increase it by 1 the corresponding increase in $2x + 1$ is always 2, we see that the graph has the same slope everywhere, which proves that the graph must be a straight line. In the same way we see that the graph of any expression of the first degree is a straight line and so such an expression is called a *linear* expression.

So far y need not be introduced; but now it should come in and we can proceed to solve a pair of simultaneous equations by means of their graphs; but the idea should be developed with some care, otherwise a mere rule will be obtained. See G. and S., *Algebra*, § 71.

After that the class will be amused at solving one or two pairs of simultaneous linear equations, but care should be taken to avoid any pretence that a graphical method is as good as the algebraical method.

After such a course as I have sketched above, graphs can come in naturally in the course of other work, e.g. when solving quadratic equations, a graphical solution is instructive; later on harder equations can be solved, and the way has been paved for calculus.

CHAPTER IX

GENERALISATION AND FORMULAE

We have already said that one of our chief aims in algebra is the idea of generalisation, and the construction and use of formulae; and in the course of the previous work we have used letters, in fact we have generalised and made and used formulae to some extent.

E.g. we have had examples in which we have seen that in s shillings there are $12s$ pence, and in p pence there are $\frac{p}{12}$ shillings.

In h hours walking at 3 miles an hour, the distance walked is $3h$ miles, etc., etc.

We want to develop these ideas a little more and we have at our command now the index notation, so that we can use more than a single letter to the first power. We want to lay more stress on the idea of generalisation. It is the mark of the mathematician, as opposed to the mere arithmetician, that, if he has to work out some long calculations, he likes to work it out in letters, in fact get a formula, and then put in the numerical values.

Changes of units afford us many examples which the teacher can make up out of his head.

How many yards are there in a mile? In two miles? In three miles?

How would you find the number of yards in any number of miles? Give the answer in words.

How many yards are there in m miles? Notice how much shorter it is to say m miles $= 1760m$ yards than to express it in words.

Already in many problems the boy has applied the idea of using letters for a number and has in fact generalised and made formulae.

Now he may have further practice in such examples and in reducing a "wordy" statement to symbols, to a formula; and, conversely, being given a symbolical statement, he can express the same fact in words.

Also, it is natural now to evaluate various expressions for given values of the letters involved. Naturally such examples are better if the meaning behind the formula is expressed; the child then sees the object of the work, but no doubt some examples will be given in which mere evaluation is required; still, the child will appreciate the value of these as the appetite will have been stimulated by the other examples given.

CHAPTER X

MULTIPLICATION, DIVISION, FACTORISATION

MULTIPLICATION

One of the features of the old teaching of algebra was the purely mechanical grind at the four rules that came at the very beginning of the course. Today we lay much less stress on the four rules: they are not done till the need for them arises, and long complicated examples may just as well be avoided altogether, thus time is saved for more fruitful topics.

It is of much more importance for a boy to be able by inspection to remove the brackets from such expressions as $x(x-3)$ and $(x-2)(2x+3)$ than to be able to do a long multiplication sum. If only boys were drilled at expanding expressions such as $(x-2)(x+5)$ and $(3x-2)(4x-3)$, there would be little trouble in teaching them to factorise trinomial expressions. It is far too common to find that boys expand such expressions by setting them out as long multiplication sums; what is the hope of such boys learning to factorise trinomials with ease and speed?

Probably by this stage a well-taught class will have the desire to expand such expressions as I have given above; but in any case it is useful to draw a rectangle with sides $(x+2)$ inches and $(x+5)$ inches, say, and split it up into its component parts; then go on to a rectangle with sides $(a+b)$ inches and $(c+d)$ inches.

From such examples, it is easy to see that to expand $(a+b)(c+d)$, the $(a+b)$ has to be multiplied by the c and then by the d and the two results added.

Now let us apply this to $(3x-2)(4x-5)$. To begin with, the boy should read off $12x^2-8x-15x+10$. After one or two examples read off in that way, the teacher, as soon as the boy

has said $-8x - 15x$, should say "that is" and the boy should say $-23x$. The boy should do a few examples with the teacher saying "that is," and then the boy should say it himself. Soon he will merely read off the answer, saying to himself "$-8x - 15x$, that is $-23x$"; but it is important not to hurry to this stage, otherwise he will develop tricks and cease to think of the full way of doing it.

The factorisation of trinomial expressions should now present little difficulty and should be taken at once. Take $2x^2 + 5x + 3$. The boy knows that he has met this as the result of expanding $(\quad)(\quad)$. He sees at once that it must be $(2x + \quad)(x + \quad)$; and he will easily see that the last terms are to be 3 and 1. All that is necessary is to try whether $(2x + 3)(x + 1)$ or $(2x + 1)(x + 3)$ gives the right result.

It is useful to put out the two possibilities thus $(2x + {}^{3}_{1})$ $(x + {}^{1}_{3})$ and cross off the one that does not work.

If this work is presented to the class as a mere puzzle (given the answer to a multiplication sum, what was the question?), they will treat it as an amusement and will acquire considerable skill before they realise that it is a useful process that they must learn.

I want to lay stress on the fact that it is wiser to start with examples in which the coefficient of x^2 is not 1, but a number such as 2, 3, 5, 7 that is prime*.

Of course, in connection with factorising trinomials it is necessary to point out that, if the absolute term is positive, the signs in the brackets will be both + or both − and that, if the absolute term is negative, the signs in the brackets will be one + and the other −. This should not be given as a rule, but gradually drawn out from the class.

* If examples such as $x^2 + 7x + 10$ are taken first, the boy may get a rule (which used to be taught) "find two numbers whose product is 10 and whose sum is 7." This is pernicious because the rule is unnecessary and it actually obscures the general method.

But to return to multiplication, at this stage the boy should learn to write down at sight the squares of expressions such as $3x + 2$ and $5x - 3y$, and he must be drilled till he is safe at this. $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$ may either be led up to by considering the figures given in most geometry books, or they may be obtained by the rules of algebra, and the geometrical figures used afterwards as illustrations; in either case here is an opportunity of linking up algebra and geometry.

It is very instructive at this stage to get the class to find the coefficient of x^2, say, in the expansion of

$$(x^3 - 7x^2 + 3x - 2)(2x - 3),$$

or $$(x^3 - 7x^2 + 3x - 2)(4x^2 + 2x - 3).$$

They do not want to do a long multiplication sum, but they should be able to pick out the terms in the two brackets that must be taken together to get x^2.

Long multiplication should really be quite unnecessary; two or three examples may be done as an illustration of neat arrangement and to bring out the meaning of long multiplication in arithmetic. It is instructive to multiply $(t^3 + 2t^2 + t + 3)$ by $(2t^2 + t + 1)$ and 1213 by 211 and to compare the two processes; then, to show that the analogy is not complete, multiply $(3t^2 + 4t + 2)$ by $(2t + 7)$ and 342 by 27. In the case of the latter pair of sums, the numerical result can be deduced from the algebraical result by putting $t = 10$, but in the arithmetic there is the complication of carrying from column to column.

DIVISION

The class must have a little *vivâ voce* practice in division by a monomial and even in examples such as

divide $3x^2(x + 2)^2 - 5x(x + 2) + 7(x + 2)$ by $(x + 2)$.

This will help with factors later.

In long division all that is wanted is to bring out the analogy

between the algebraical process and the arithmetical, and there is no need for more than a few examples. Examples such as divide $(a^3 + b^3 + c^3 - 3abc)$ by $(a + b + c)$ should be avoided entirely.

FACTORISATION

I always tell boys that in factorising any expression the first thing is to see whether there is a factor that shouts at them. If it does not shout at the boy, his eye must be trained until it does.

Thus, in $4y^2 - 6xy$, $2y$ should shout at you, it occurs in every term. There is no difficulty about that type; the other factor is obtained by dividing each term by $2y$.

It is not much harder to deal with expressions such as $2x(x + 3) - 5(x + 3)$. Here $(x + 3)$ shouts at you, so the expression equals $(x + 3)(2x - 5)$.

Strange to say, if a boy is asked in an examination such as the Common Entrance to factorise $2x(x + 3) - 5(x + 3)$, he almost invariably multiplies out and gets $2x^2 + x - 15$ and proceeds to factorise that. This shows that there is need to lay stress on the "shouting" type of factor.

There is the type of expression in which we can group terms together so that we get "shouting factors." A useful tip here is to spot some letter that occurs only in two terms and to group those terms together. Thus, in $ab + cd + ac + bd$, a occurs in two terms and those terms give us $a(b + c)$, the other two terms give us $d(b + c)$, therefore the expression equals $(b + c)(a + d)$.

Again, consider $a^2 - bc - ab + ac$, b only occurs in two terms. Take them together.

$$\begin{aligned}\text{The expression } &= -b(c + a) + a^2 + ac \\ &= -b(c + a) + a(a + c) = (a + c)(a - b).\end{aligned}$$

The difference of two squares gives another opportunity of going into geometry. From a square sheet of paper of side a cut away a square of side b at one corner, divide what remains into two rectangles and fit them together. It sounds difficult, but look it up in the geometry book; and it is worth having a model cut out of cardboard.

The actual examples present little difficulty except in cases in which the class has to group three terms together to make a square, and that should not be hard if they really know $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

I am afraid my standard way of driving the result home is $(\text{chair})^2 - (\text{table})^2 = (\text{chair} + \text{table})\,(\text{chair} - \text{table})$ and I ask "what is the chair?" "what is the table?"

Of trinomials I have spoken in connection with multiplication.

When boys have become skilful at them there are some refinements that are worth noticing.

E.g. $4x^2 - 3x \ldots$ must be $(4x \ldots)\,(x \ldots)$.

$(2x \ldots)\,(2x \ldots)$ would give an even number of x's on expansion.

Again, $4x^2 +$ an even number of x's must be $(2x \ldots)\,(2x \ldots)$ unless the last term is even, in which case there is a "shouting" numerical factor

At this stage there is no need for any further factors.

$a^3 \pm b^3$, $a^3 + b^3 + c^3 - 3abc$, etc., all belong to the specialist stage.

CHAPTER XI

QUADRATIC EQUATIONS

The importance of quadratic equations has perhaps been exaggerated in the past, but there are at least two valuable ideas to be gained from their solution:

(i) There is some novelty about solving an equation which contains x^2.

(ii) There is certainly novelty in finding that an equation can be true for more than one value of x.

There are two natural methods of approaching the subject; either we may tackle a problem that leads to an equation involving x^2, or we may consider a graph such as $x^2 - 4x + 1$ and notice that there are two values of x for which $x^2 - 4x + 1$ is equal to zero.

SOLUTION BY FACTORS

First of all boys will be taught to solve quadratics by factors. The point to drive home is that if the product of two or more factors is 0, one of the factors must be 0. The class will raise objections but they are easily dealt with.

To solve $3 + 5x = 2x^2$.

The work will be put out thus:

$$3 + 5x = 2x^2,$$
$$\therefore\ 2x^2 - 5x - 3 = 0,$$
$$\therefore\ (2x + 1)(x - 3) = 0,$$
$$\therefore\ 2x + 1 = 0 \text{ or } x - 3 = 0,$$
$$\therefore\ 2x = -1 \text{ or } x = 3,$$
$$\therefore\ x = -\tfrac{1}{2} \text{ or } 3.$$

In a few examples the results should be checked.

This piece of work is done so easily and quickly, and the more general method takes so much longer, that the factor method is apt to be forgotten. To avoid this, I would suggest that the teacher should keep on harping on the fact that the factor method should always be tried first.

GRAPHICAL SOLUTION. I

The class will realise the limitations of solution by factors, and it is nice to suggest a graphical solution so as to prepare them for the idea that an approximate solution is possible for any equation when algebraical methods fail (see chap. XIV).

They should notice that, if the graph of $y = x^2 + 3x - 2$ is drawn, they can obtain from it the solutions not only of $x^2 + 3x - 2 = 0$, but also of $x^2 + 3x - 2 =$ any number, and so of $x^2 + 3x =$ any number.

But a graphical solution is only approximate so we must go on to

SOLUTION BY COMPLETING THE SQUARE

A week or two before tackling this the teacher who looks ahead will have made sure that the class are quite certain that

$$(a + b)^2 = a^2 + 2ab + b^2 \text{ and } (a - b)^2 = a^2 - 2ab + b^2,$$

and can apply these. As a little extra amusement he may have asked the class to fill up the gaps in such cases as

$$x^2 + 6x + 9 = (\quad)^2,\ x^2 - \quad + 25 = (\quad)^2,$$
$$x^2 - 4x + \quad = (\quad)^2 \text{ and even } 4x^2 + 12x + \quad = (\quad)^2,$$

but this work need not have been very systematic at that stage.

First let us consider $x^2 = 9$; the class will at once suggest $x = 3$ as a solution, but let us also solve the equation by factors and we get $(x - 3)(x + 3) = 0$, $x = 3$ or $- 3$. Test both these results and we see that 3 and $- 3$ are both solutions.

Similarly solve $x^2 = 16$, $(x + 2)^2 = 25$. A little *vivâ voce* work of that type and a couple of written examples will clear away one of the difficulties we shall encounter.

The next thing is just a couple of minutes for refreshing their memories by expanding $(x + 2)^2$, $(x - 5)^2$; do not avoid§ $(2x + 3)^2$.

Now let us tackle $x^2 + 6x = 27$. Just for once we will depart from our rule and not bother to see whether it can be done by factors.

$x^2 + 6x + \quad = (x + *)^2$. Here we have two gaps; what number shall we put where the * is? Someone is pretty sure to suggest 6. "But $(x + 6)^2$ starts with $x^2 + 12x$. Guess again." A few examples will get the majority guessing right every time; but what are we to do with those who cannot do it? Put an a where the * is, then $x^2 + 2ax$ has to be the same as $x^2 + 6x$, $\therefore 2a = 6$, $\therefore a = 3$†.

Therefore we have the following:

$$x^2 + 6x + \quad = 27 + \quad .$$

The square when completed is $(x + 3)^2 = x^2 + 6x + 9$,

$$\therefore x^2 + 6x + 9 = 27 + 9,$$

$$\therefore (x + 3)^2 = 36$$

(which is a type with which we are already familiar),

$$\therefore x + 3 = 6 \text{ or } - 6,‡$$

$$\therefore x = 3 \text{ or } - 9.$$

§ If this is avoided, they may get a false rule, namely that the number of x's is twice the absolute term in the bracket. Lay stress on the fact that the middle term is twice the product of the two terms in the bracket.

† As a slight modification that will help some boys, try the following—it is the same as this only in words. We get the middle term by taking twice the number we are going to put for the * times x; what must I put? Or even, twice what make 6?

‡ This is better than ± 6 for a few days.

Check. When $x = 3$, L.H.S. $= 9 + 18 = 27 =$ R.H.S.

When $x = -9$, L.H.S. $= 81 - 54 = 27 =$ R.H.S.

The next thing is for the class to try a few examples such as in G. and S., *Algebra*, Ex. 11 *f*.

Always insist on trying factors first or the square root method will be regarded as the only method, whereas the factor method should always be used, if possible, on the ground that it is easier and less liable to mistakes.

The class will now need drill of the type $x^2 + 8x$ is the beginning of the square of ($x +$ what?). $x + 4$. And $(x + 4)^2 = x^2 + 8x + 16$ (see G. and S., *Algebra*, Ex. 11 *e*). This drill produces better results *after* they have done a few questions from Ex. 11 *f*, mainly, I suppose, because they see the use of it then, whereas before they do not see what they are driving at.

I should like to preach a long sermon against giving the rule "add on the square of half the coefficient of x," but I will be brief. The main objection is that it is a *rule* and it is perfectly easy to use reason instead, as suggested above. I know the reasoning behind the rule can be explained and understood and the boy will probably invent it himself all in good time; but the fact remains that, if the rule is given prematurely, it is applied unintelligently and in nearly all cases it is regarded as a sort of trick or charm. Ask a class to solve $4x^2 + 20x = 7$, without first dividing through by 4. If they have been given the rule, the majority will add 100 to each side; whereas, if they have been taught to write down the square and square out, the majority will do it correctly; and further, it is hard to convince the "rule" boy that he is wrong but the "non-rule" boy, if he makes a mistake, is convinced of it at once.

As to solving a quadratic such as $6x^2 - 10x + 3 = 0$, there is little to say. It is best to divide both sides by 6 and the work appears thus:

$$6x^2 - 10x + 3 = 0,$$

$$x^2 - \tfrac{5}{3}x = -\tfrac{1}{2}.$$

The square on the L.H.S. will be $(x \quad \frac{}{6})^2$,

$$\therefore x^2 - \tfrac{5}{3}x + (\tfrac{5}{6})^2 = -\tfrac{1}{2} + \tfrac{25}{36},$$

$$\therefore (x - \tfrac{5}{6}) \qquad \tfrac{}{36}.$$

Take the square root of each side,

$$\therefore x - \frac{5}{6} = \pm \frac{\sqrt{7}}{6},$$

$$\therefore x = \frac{5 \pm \sqrt{7}}{6} = \frac{5 \pm 2{\cdot}646}{6}$$

$$= \frac{7{\cdot}646}{6} \text{ or } \frac{2{\cdot}354}{6}$$

$$= 1{\cdot}27 \text{ or } 0{\cdot}39.$$

If in the course of solving a quadratic equation we find say $(x - 1)^2 =$ a negative number, it is enough at this stage to say that $x - 1 =$ an unintelligible number, and therefore we can find no intelligible value for x that satisfies the equation. This can be well illustrated from a graphical solution (see p. 204).

Some reader is sure to ask whether boys should learn to solve quadratics by formula. My opinion is that no boy should be introduced to the use of the formula for at least a couple of years after he has learnt the square root method. There is no necessity for it earlier: it is not a great saving of labour, and an early introduction of it seems to drive out of a boy's head the method without formula and various little bits of manipulation that are of great use in other work later.

Some teachers favour using the method indicated below for solving quadratics involving square roots. The argument put forward in favour of it is that all quadratics are then solved by

factors; but in practice boys certainly find it harder than the method given on pp. 206, 207.

$$6x^2 - 10x + 3 = 0,$$

$$\therefore \quad x^2 - \tfrac{5}{3}x + \tfrac{1}{2} = 0,$$

$$\therefore \quad (x - \tfrac{5}{6})^2 - (\tfrac{5}{6})^2 + \tfrac{1}{2} = 0,$$

$$\therefore \quad (x - \tfrac{5}{6})^2 - \tfrac{7}{36} = 0,$$

$$\therefore \quad \left(x - \frac{5}{6}\right)^2 - \left(\frac{\sqrt{7}}{6}\right)^2 = 0,$$

$$\left(x - \frac{5}{6} + \frac{\sqrt{7}}{6}\right)\left(x - \frac{5}{6} - \frac{\sqrt{7}}{6}\right) = 0,$$

$$x - \frac{5}{6} + \frac{\sqrt{7}}{6} = 0 \text{ or } x - \frac{5}{6} - \frac{\sqrt{7}}{6} = 0,$$

....

GRAPHICAL SOLUTION. II

The class will have already learnt that they can solve a quadratic equation such as $2x^2 - 5x - 1 = 0$ by drawing the graph $y = 2x^2 - 5x - 1$ and that from this graph they can solve $2x^2 - 5x =$ any number.

Now they may well learn that if they draw $y = x^2$, they can solve any quadratic by drawing the appropriate straight line across it. Thus

$$2x^2 - 5x - 1 = 0,$$

$$x^2 = \frac{5x + 1}{2}.$$

So draw the line $$y = \frac{5x + 1}{2}.$$

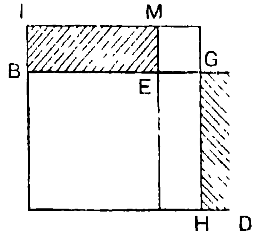
NOTE

A nice geometrical illustration of the solution of a quadratic can be obtained as follows:

Suppose we want to solve $x^2 + 6x = 50$.

ABCD is a rectangle whose area is 50 sq. in., **AB** $= x$ in., **AD** $= (x+6)$ in.; **ABEF** is x^2 sq. in.

Divide **FECD** into two equal parts and move **GHDC** to the position **BEMN**. Produce **NM** and **HG** to meet at **P**.



Then

$$\text{AHPN} = \text{ABCD} + \text{EGPM},$$

i e. $\qquad (x+3)^2 = 50 + 3^2.$

SIMULTANEOUS QUADRATICS

Though they will be done later, this is perhaps a convenient place to refer to simultaneous quadratics. The only case that ought to be considered as within the range of the non-specialist is that in which one equation is linear and one quadratic. That case can always be solved by using the linear equation to express one of the unknowns in terms of the other, and then substituting in the quadratic.

It is always useful with a class to go through a whole set of simultaneous quadratics of the type mentioned (see G. and S., *Algebra*, Ex. 18*a*), merely discussing in each question whether it is better to substitute for x or for y.

One other point needs discussion, namely why it is necessary, after finding the values of one unknown, to substitute in the linear equation and not in the quadratic. This must be considered graphically (see G. and S., *Algebra*, § 177).

CHAPTER XII

TRANSFORMATION OF FORMULAE LITERAL EQUATIONS

In the algebra books of the end of the last century literal equations were introduced far too early: it was not uncommon to find literal equations occurring in the first set of examples on the solution of equations. Some teachers did not appreciate the gap to the boy between solving, say,

$$\left.\begin{aligned} ax - by &= a^2 - b^2 \\ bx + ay &= 2ab \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} 3x - 2y &= 5 \\ 2x + 3y &= 12 \end{aligned}\right\};$$

even the long grind he had had at meaningless manipulation did not in many cases give him the skill to deal with the first pair of equations. He did not appreciate the aim of such work.

Today literal equations arise naturally out of the manipulation of a formula. He is given the formula $V = \pi r^2 h$, from which it is easy to find the volume of a cylinder when given the radius and the height. He can see that it is easy to find the height when he is given the volume and the radius (or the radius when he is given the volume and the height); if he has a single example to do, no doubt it is simpler to substitute the numbers in $V = \pi r^2 h$ and do the manipulation after that; but manipulation with letters will be necessary later, and the boy can see that, if he had to substitute many different sets of numerical values, it would be more convenient to turn the formula round into the form $h = \dfrac{V}{\pi r^2}$ or $r = \sqrt{\dfrac{V}{\pi h}}$, and then to evaluate it; in other words it would be more convenient first to change the subject (or the nominative) of the sentence.

The more of a mathematician the boy becomes the more he

will tend to manipulate with letters and keep out numbers until the end.

Too much stress cannot be laid on the fact that $V = \pi r^2 h$ is merely shorthand for a sentence; which is the verb? which the subject (or nominative) of the sentence?

Treated as I have suggested, the subject has meaning and becomes attractive. But much practice must be given and many difficulties will arise; these will have to be treated kindly and we shall constantly have to go back to numerical equations and consider more carefully what our method of procedure really was there.

E.g. suppose that we are given

$$t = \sqrt{\frac{P}{P - W} \times \frac{h}{16}},$$

and are asked to express P in terms of the other letters involved

$$t^2 = \frac{hP}{16P - 16W},$$

$$\therefore\ t^2\,(16P - 16W) = hP.$$

So far there is no difficulty, but there will probably be a check here.

What is really the unknown here? It is P. Let us underline P in our last equation

$$t^2\,(16\underline{P} - 16W) = h\underline{P} \qquad \ldots\ldots\text{(i)}.$$

Now what should we do with an equation

$$9\,(16P - 80) = 6P \qquad \ldots\ldots\text{(ii)}?$$

We should get all the terms containing P on one side of the equation and all the other terms on the other side. Then let us do the same here with (i) and (ii):

$$\text{(i)}\quad 16t^2\underline{P} - 16t^2W = h\underline{P},$$

$$\therefore\ 16t^2\underline{P} - h\underline{P} = 16t^2W \qquad \ldots\ldots\text{(iii)}.$$

$$\text{(ii)}\quad 144P - 720 = 6P,$$

$$\therefore\ 144P - 6P = 720 \qquad \ldots\text{(iv)}.$$

What do we do with (iv)? We say $(144 - 6) P = 720$.

Then with (iii), $(16t^2 - h) P = 16t^2 W$

Obviously now we have only to divide both sides by $(16t^2 - h)$

$$P = \frac{16t^2 W}{16t^2 - h}.$$

Many similar difficulties will arise: they must each of them be treated in the same sort of way by considering what we did in numerical cases.

After struggling with many examples like this, I like to give boys a plain set of literal equations, e.g. G. and S., *Algebra*, Miscellaneous Exercises, p. 249, No. 143. Some teachers would give this first and then go back to Ex. 12 *a*, but I am convinced that the solution of literal equations has more meaning for the boy after he has done some transformation of formulae. Such work throws much light on how the boy was taught in the early stages of algebra; a boy who was well taught will soon be able to solve literal equations, but the boy who has learnt mechanically is very slow at them, and it is hard work because he has not realised fully what he was doing in solving equations with numbers in them.

CHAPTER XIII

FRACTIONS

Much of the time devoted to algebra at the beginning of this century was taken up with long complicated fractions, many of them so complicated that none but specialists were ever likely to come across such fractions in any real work which they would do; the examples were invented purely and simply for the sake of the manipulation, and that not useful manipulation. Such questions appeared regularly in examination papers so that teachers were driven to grind at these unfruitful questions; the effect on the pupils was to dull their interest and to convince them that algebra was merely meaningless manipulation

Happily such questions no longer appear in Certificate examination papers. This enables fractions to be dealt with more intelligently and in much less time than formerly.

If the work with vulgar fractions in arithmetic and with algebraical fractions with numerical denominators has been well done, there are no new principles to be introduced and the work is mainly drill.

One special warning is necessary when dealing with equations in which fractions occur.

Suppose that in an equation there is a term $\frac{2x+5}{x-3}$. If you multiply both sides by $x-3$, you may introduce a root $x=3$; so it is wise to test the roots of such an equation. If $x=3$, it should be clear that $\frac{2x+5}{x-3}$ is meaningless.

Ask a class to solve the equation

$$\frac{2}{x-1}+\frac{2x+1}{x-2}=\frac{3x-1}{x-1}.$$

You will probably find that many boys give a root $x=1$.

CHAPTER XIV

EQUATIONS OF HIGHER DEGREE THAN THE SECOND

So far as the algebraical solution of equations is concerned the class already know how to solve equations (in one unknown) of the first and second degrees; and they can be told that it is possible, though not easy, to solve equations of the third and fourth degrees, but that it is not possible to solve equations of the fifth or higher degree.

They will be interested to see that any equation of whatever degree can be solved approximately by means of a graph, and they should solve a few equations in that way, for example, some cubics. How much work they should do at this will depend on the time they have to spare for it and on their ability, but they should at least see that

(i) if they draw the graph $y = 2x^3 - 10x - 7$, they can solve any of the family $2x^3 - 10x =$ any number, and that any such equation has either one or three real roots;

(ii) if they draw $y = x^3$ and a line $y = ax + b$ (where a and b are any numbers), the x's of the point or points of intersection are roots of the equation $x^3 = ax + b$, so that we can solve any cubic of that type.

That is as much as I should attempt as a rule, but with a very bright class I should not hesitate, if time permitted, to show how any cubic can be reduced to the form $x^3 = ax + b$, so that method (ii) above enables us to solve any cubic in a shorter way than method (i) would.

CHAPTER XV

INDICES AND LOGARITHMS

When leading up to logarithms the teacher has a great opportunity of showing how mathematics has grown in the past and is still growing at the present time.

Consider how the race's idea of numbers has grown: (i) the positive integers 1, 2, 3, etc., (ii) the idea of fractions, then, after a long interval, (iii) the idea of negative numbers.

Now let us consider 10^x; this has a definite meaning so long as x is a positive integer, and it is easy to prove that

$$10^x \times 10^y = 10^{x+y}$$

(if x and y are positive integers)—I shall refer to this as the great index law. Now, whatever meaning 10^x may have, if x is not a positive integer, it is desirable that 10^x should still obey the same laws as it obeyed when x was a positive integer.

Let us experiment with $10^{\frac{1}{2}}$. If we assume the great index law

$$10^{\frac{1}{2}} \times 10^{\frac{1}{2}} = 10^{\frac{1}{2}+\frac{1}{2}} = 10^1 = 10,$$

$$\therefore\ 10^{\frac{1}{2}} = \sqrt{10}, \text{ and } \sqrt{10} \text{ has a meaning.}$$

Similarly we can take

$$10^{\frac{1}{3}} \times 10^{\frac{1}{3}} \times 10^{\frac{1}{3}} = 10^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 10^1 = 10,$$

$$\therefore\ 10^{\frac{1}{3}} = \sqrt[3]{10}.$$

From these and similar examples we can get a meaning for 10^x when x is a positive fraction.

Also $10^0 \times 10^7 = 10^{0+7} = 10^7$. Hence $10^0 = 1$.

Note what we have done here. 10^x had originally a meaning for a limited field of values of x (viz positive integral values of x). We have now extended the meaning of 10^x over a new field of values of x, taking care that the extended meaning does

not contradict the original meaning. This sort of process goes on in all sorts of places in mathematics, e.g. in trigonometry: the sine of an angle is originally defined for a positive acute angle, soon the necessity arises for extending the meaning of $\sin a$ where a is no longer a positive acute angle, and then we extend our definition, but take care that our extended definition does not contradict the original definition.

It is well to let the boy see what a wide idea we have here, even though he cannot at present appreciate how wide it is. But we must return to logarithms.

It is instructive to show a boy how, by finding

$$10^{\frac{1}{2}} = \sqrt{10} = 3{\cdot}162,$$

and taking the square root again

$$10^{\frac{1}{4}} = \sqrt{3{\cdot}162} = 1{\cdot}779,$$

he can construct a table of the indices of 10 corresponding to a lot of numbers; from that he can go on to graph them and see that with more and more labour he could go on to greater and greater accuracy. But we need not waste time over this. Napier and Briggs did this for us 300 years ago* and their tables are the basis of all modern tables.

Then we may turn over to the tables and verify the results we have worked out, e.g. the number 3·16 has as corresponding index ·4997, or $3{\cdot}16 = 10^{\cdot 4997}$ (we found $3{\cdot}162 = 10^{\cdot 5}$), etc.

We must now teach the boy how to use the table, first how to look up the index for a number of three significant figures and then for a number of four significant figures. Then we may do a few carefully selected multiplication and division sums (selected so that only numbers between 1 and 10 are involved).

* It interests boys to know that the tercentenary celebrations of the invention of logarithms were being held in Edinburgh at the time the Great War broke out. I believe that a French professor was actually lecturing at the celebrations when a telegram was handed to him, he read it and said, "France is mobilising, I must go."

He will at once want to go on to other numbers and he will follow at once that

$$31{\cdot}62 = 10 \times 3{\cdot}162 = 10^1 \times 10^{\cdot 5}$$
$$= 10^{1\cdot 5},$$

and similarly with larger numbers.

Now he can do many multiplication and division sums and it is wise that he should set them out using the index notation. Thus

$$\begin{aligned}
25{\cdot}37 \times 457{\cdot}6 &= 10 \times 2{\cdot}537 \times 10^2 \times 4{\cdot}576 \\
&= 10^1 \times 10^{\cdot 4043} \times 10^2 \times 10^{\cdot 6605} \\
&= 10^{1\cdot 4043 + 2\cdot 6605} \\
&= 10^{4\cdot 0648} \\
&= 10^4 \times 10^{\cdot 0648} \\
&= 10^4 \times 1{\cdot}161 \\
&= 11610.
\end{aligned}$$

Soon he will shorten it to

$$\begin{aligned}
25{\cdot}37 \times 457{\cdot}6 &= 10^{1\cdot 4043} \times 10^{2\cdot 6605} \\
&= 10^{1\cdot 4043 + 2\cdot 6605} \\
&= 10^{4\cdot 0648} \\
&= 11610.
\end{aligned}$$

Two or three points arise out of this:

(i) A number in standard form (e.g. 3·142), i.e. a number between 1 and 10 (i.e. between 10^0 and 10^1), has an index between 0 and 1, that is nought point something.

(ii) The whole number in the index can be found by counting how many places the decimal point is away from the standard place.

I have found it useful to teach boys at first to put an arrowhead at the standard place.

Thus
$$2\overset{\downarrow}{\,}5{\cdot}37 = 10^{1\cdot 4043},$$
$$10^{4\cdot 0643} = 1\overset{\downarrow}{\,}1610.$$

They soon get beyond the stage at which the arrow is helpful and then they drop it.

I never introduce the words "mantissa" and "characteristic."

Also, when a boy writes say $3 = 10^{0 \cdot 4771}$, I always insist on his putting the 0 in front of the decimal point; I say that, if he merely leaves a blank, I shall assume that he has not considered what to put there.

So far, it may be noted, I have always spoken of "the index corresponding to the given number"; gradually the boy may be introduced to the alternative expression "the logarithm of the given number"; quite soon he will drop the former phrase and will always speak of the logarithm of 7·2 (say) or log 7·2.

I feel very strongly that it is unwise to let boys use anti-logarithm tables for the first year in which they use logarithms. If they use both tables, my experience is that they regard the whole thing much more as jugglery; but, if they use only the log table, it is easier to keep before them the idea that the figures round the edge of the table are the numbers and the figures in the body of the table are the corresponding indices or logarithms.

Again, some day the boy must learn to use a table backwards so to speak (we have no anti-sine tables), so he may as well learn now.

After going so far with logarithms, I have found it a good thing to give the subject a thorough rest, at least two or three weeks. Then I think it is wise to go rapidly over the whole ground again, and then to extend the work to include negative indices. There is no need here to go fully into the question of negative indices, but I should like to draw attention to a few things.

Many of us were taught an absurd rule in the days of our youth, about the whole number in an index (or logarithm) being one less than the number of figures to the left of the decimal point or one more than the number of 0's after the point. A

wicked rule: the boy always forgot which was one more and which one less; but more wicked because it is unnecessary and hides the reasoning. It is far simpler to think of the number of places that the point is away from the standard place—there is reason in that, it shows the power of 10 by which the standard form number has to be multiplied or divided to get the given number, and so the whole number that has to be added to or subtracted from the fractional part of the index or logarithm.

Some practice must be given in adding and subtracting indices such as $\bar{1}\cdot6211$ and $\bar{2}\cdot5321$. This is best done with *vivâ voce* examples. See G. and S., *Algebra*, Ex. 15*i*.

My own practice is to tell the boy to get the whole number in the index by writing it in the margin.

Thus to add

$3\cdot7$	First put down 1 (the figure carried) then the
$\bar{2}\cdot9$	3 and the -2. Thus $1+3-2=2$.
$2\cdot6$	

In subtracting

$3\cdot7$	First put down -1 (borrowed or whatever he
$\bar{2}\cdot9$	calls it); then 3 and subtract the -2. Thus
$4\cdot8$	$-1+3-(-2)=-1+3+2=4$.

My own experience leads me to suggest keeping to the use of index notation (see p. 217) for the first term in which a boy does logarithms. After that use tabular form thus:

No. Log

But a boy should always be able to put out his work in index form if asked to do so.

ACCURACY OF FOUR-FIGURE TABLES

It is important to impress on boys that when working with four-figure logarithms the fourth significant figure in the answer can only be approximate, for

(i) the fourth figure as printed is only the nearest figure, so that when adding several logarithms the fourth figure is subject to a small error;

(ii) the figures given in the difference columns are only average differences and may be even as much as 2 out.

Consequently, where differences are inclined to be untrustworthy (e.g. near the beginning of the logarithm table), it is better not to use the difference columns, but to interpolate for oneself.

Also it is better in looking up log 2789 to look up log 279 and subtract the difference for 1 than to look up log 278 and add the difference for 9, and similarly for 6, 7, 8.

LOGARITHM NOTATION

I do not consider that there is any need to introduce the boy to this notation for a year or two after he has begun to use logarithms, but then he should be shown it and understand

$$\begin{aligned}\log\{27{\cdot}63 \times 13{\cdot}64\} &= \log 27{\cdot}63 + \log 13{\cdot}64,\\ &= 1{\cdot}4414\\ &+ 1{\cdot}1348\\ &= 2{\cdot}5762,\\ \therefore\ 27{\cdot}63 \times 13{\cdot}64 &= 376{\cdot}9.\end{aligned}$$

CHAPTER XVI

VARIATION

In the older text-books the work on variation seems to be mere k-juggling, and most of the modern books seem to make but little of the subject, though it seems to deserve a fuller and deeper treatment. Many teachers seem to regard it as a side issue, but it seems to be a most important main-line subject.

Variation should be treated from a "functionality" point of view and should be intimately associated with graphs; it really amounts to little more than the study of graphs and functions with the addition of a little verbal nomenclature.

It should be correlated with physics; teachers of physics often complain that boys come to them with no grasp of the idea "varies as." This is due to boys being allowed to take refuge in k-juggling before they have soaked themselves in the functional idea of variation.

The more algebraical part of the subject, and especially the determination of constants, should not enter till the idea of different forms of variation is as familiar as the idea of multiplication. The first step may be the consideration of a set of related values, such as loads and consequent extensions of a spring. These values will naturally be subject to the errors of experiment; purely arithmetical methods might not reveal the law, but plotting the values will suggest it; we find a relation that may be expressed naively as "If you double x, you double y, etc." More precise statement, verbal and algebraical, will be worked out step by step; numerous other physical and geometrical illustrations are available; and gradually the pupil should attain a conception of the simplest case of variation, the complete conception involving (i) a knowledge of the arith-

metical relation between two pairs of values; (ii) a mental picture of the graphical appearance of this case of functionality; (iii) an associated knowledge of the algebraical form of the function.

In a similar way, the other common forms of variational functionality will be studied one at a time. In each case the instruction is incomplete till the pupil can say without hesitation, "In this kind of variation, if you double x, you double (quadruple, halve, quarter, etc.) y"; it is incomplete till the graphical and algebraical forms are so firmly associated as to suggest one another.

With variation may be studied such common forms of functional relationship as $y = ax + b$; strictly not a case of variation in the usual sense of the word, but in its essence forming part of the same chapter of thought.

CHAPTER XVII

FURTHER ALGEBRA

The work which I have considered in algebra is quite enough for the ordinary pupil: if he has covered that carefully, has a grasp of the ideas involved and has reasonable skill with the manipulation he has met, he has gained all that he wants from elementary algebra. There are a few frills that he might add, a little simple work on surds and ratio and proportion perhaps. So far as algebra is concerned, he might do a little work at progressions and so get the idea of a series, but it is better for him to go on to elementary calculus which is much more fruitful in ideas and will give him a sense of power. (See "Further Mathematics," p. 315.)

One difficulty a teacher always experiences is that two or three of the abler members of a class are ready for new work before the rest have consolidated the work in hand. Probably those who are ready to go ahead will later on become mathematical or science specialists; if the master maps out a course for them, they can go on with it when they have time on their hands. The master will probably not have much time for helping with this extra work; for this and other reasons, it is probably better that it should not go on to entirely new work that will be met when the boys become specialists, but should include side topics of the range already covered: harder fractions and equations might be done, elementary theory of quadratic equations, further graphical work, harder manipulation and miscellaneous examples generally.

PART V

GEOMETRY

BY A. W. SIDDONS

The numbers given to theorems are those that occur in Siddons and Hughes' *Junior Geometry* (S. and H., *J.G.*) and in the same authors' *Practical Geometry* (S. and H., *P.G.*) and *Theoretical Geometry* (S. and H., *T.G.*).

Junior Geometry follows precisely the course sketched out here up to Chapter XIV, but the same course can be taken with most of the good modern text-books.

See also Siddons and Snell's *Introduction to Geometry* and *A New Geometry*.

CHAPTER I

WITH WHAT KNOWLEDGE DOES A CHILD COME TO THE STUDY OF GEOMETRY IN SCHOOL? AT WHAT AGE SHOULD IT BE BEGUN?

If we take geometry in its literal sense, we may say that the study of the subject begins soon after birth; anyone who has watched a baby stretching out its arms and trying to touch things must realise that the child is acquiring knowledge of space. But we are not concerned with this stage.

In a secondary or preparatory school a child may well start the study of geometry at some time between the ages of 10 and 12; I should put down 11 as a good age for a child of average ability.

The work he will have done before that probably includes some very good kindergarten or Froebel work: work with toys, bricks, paper folding and sand trays; and he will have acquired a lot of geometrical knowledge from moving about in the world. He may also have had some geometry teaching of a more specific type, but in a class it is pretty certain that there will be some children whose knowledge may be summed up as follows:

They can measure in inches and tenths, in yards, feet and inches, and probably in centimetres and millimetres.

They will have acquired a certain vocabulary of geometrical words (e.g. point, square, circle, straight, long, thin, large), but much of it will be vague.

They will have some acquaintance with area, and possibly volume, from arithmetic.

They will know and recognise a *right angle*, but will probably have little knowledge of an *angle*.

They will have clear ideas of what a *straight line* looks like, and no attempt at a definition will help them.

CHAPTER II

A FIRST TERM'S GEOMETRY

The teacher's ultimate aim in teaching geometry may, I suppose, be stated briefly as follows: to get the boy to acquire a mass of geometrical facts, to see their logical relation to one another and to apply them.

The teacher's aim for the first term must be to enlarge the boy's geometrical vocabulary—get him to understand geometrical words and to use them correctly. Definitions are out of place here; they come after the boy has some knowledge of the things (see p. 255).

The boy must also develop a geometrical eye: he must learn to recognise figures and to see them not only in the class-room but out in the world.

The pupil must have an aim too. The aim to be set before him must not be some distant goal: it must be something which he can feel is almost within his reach, otherwise he will lose heart. I would suggest as a first aim the drawing of accurate plans.

First lessons. Examine models*: cubes, cuboids (a brick or rectangular block), three- and four-sided prisms, pyramids, cylinder, cone, sphere. One at a time, of course†.

Let the boy handle them and count their faces, edges and vertices ("corners" at first).

* Boxes of suitable models, containing cube, cuboid, prisms, and pyramids of 3, 4 and 6 sides, cylinder, cone, sphere can be bought quite cheaply, but boys will often like making models and the best of these can be kept and are quite good enough for our present purpose.

† For convenience I have discussed all the work with models before considering the other work of the term; but the wise teacher will not exhaust all his models at the start; a few for a stimulus at first, then the others some weeks later to give a change from scale drawing.

In a later lesson, without a model before him, ask for the number of faces, edges or corners of a brick, or a pyramid on a square base. This will make him visualise the solid.

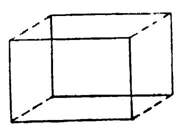
The cube. How many corners on the table? How many up above? How many edges on the table? How many on top? How many standing upright?

The pyramid. How many corners on the table? How many sloping edges?

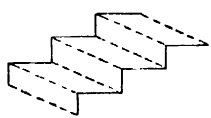
Ask for instances of the various solids that the boy sees in everyday life; the room, desks, books, tools, games, machines, etc. With a little encouragement it is surprising what a variety of instances will be suggested. This will help to get the boy to think geometrically outside the class room as well as in it.

It is a great help for the boy to learn to *draw a solid figure* (see chap. XI, p. 298); but it is a question whether there is time for this now, but it is certainly worth while to teach him to make a good drawing of a brick, as follows:

On squared paper draw a rectangle along the lines of the paper; then draw an equal rectangle slightly to the right and slightly higher up on the paper, but well overlapping the first one; now join corresponding corners (as shown by dotted lines). After drawing a few on squared paper it is good to try a few on plain paper.



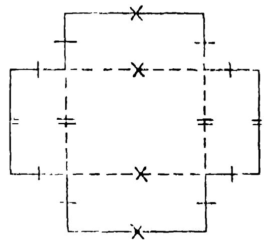
One more instance will probably encourage the boy to spend some of his spare time in making drawings of solid figures. See the figure: the two equal zigzags of continuous lines are drawn first and the dotted lines are put in afterwards; the boy will naturally draw continuous lines where the dotted lines are shown—they are only dotted here to explain the method more easily.



The making of models is a pleasant pastime, but there is probably not time for this in school or even in preparation time;

a little talk about it in school will probably produce a batch of models made in the boys' spare time.

Take the inside of a matchbox (a larger box would be better), cut it down the four upright edges and flatten it out, thus producing this figure.



Let each boy make a freehand sketch of the figure, then let the master draw it on the blackboard. Notice that all the angles are right angles.

Mark the edges that are equal, as in the figure.

Now make an accurate drawing to given measurements. Then cut the figure out and fold along the broken lines; the edges may be joined with stamp edging.

Possibly ask what figure would be obtained if a pyramid were cut down its sloping edges and flattened out. This is probably enough to stimulate several boys to amuse themselves with making models.

Tell them that if they make them in cardboard, it is best to cut the cardboard half through with a sharp knife along the line where it is to be bent.

The next thing is to develop the **idea of direction**—horizontal and vertical; learn where the North is.

Now we come to the **idea of an angle**.

It is interesting to notice that most children have a clear idea of a right angle long before they have any idea of an angle.

I have tried the following with several children:

Teacher. "Do you know what a right angle is?"

Child. "Yes."

Teacher. "Show me some."

The child has shown several, the corner of a book, a table, the ceiling.

Teacher. "Can you show me half a right angle?"

Almost invariably the answer I have got has been "That is silly, you can't have half a right angle."

From this I conclude that the child regards a right angle as a certain shape, but has no conception of an angle.

Make the children stand up and do a "right turn" (it is a help if they do this holding their arms together straight out to their front). Now do a "half right turn."

"Could you do a quarter right turn?"

Now hold your arm horizontally and move it (horizontally) through a right angle, through half a right angle, through a quarter of a right angle.

Take a clock face, or draw one on the board. Move one hand from 12 o'clock to 3 o'clock, 12 to 6 o'clock, 12 to [illegible] o'clock Through what angles has it turned?

Draw an angle on the board; then shorten the arms by rubbing out part of the arms. "Have I altered the angle? Have I made it smaller?"

Draw on the board two equal angles, one with much shorter arms than the other. Ask the question "Which is the larger angle?" Some children are pretty sure to think the angle with the longer arms is the larger.

To meet this it is useful to have two pieces of wood hinged together with a stiffish hinge, or a folding two-foot rule. Open the hinge so that the arms will fit on one of the angles, then fit it on to the other. This will help to make the class see that the size of an angle does not depend on the lengths of its arms; it will also help to give the idea of an angle as a measure of the amount of turning.

Explain that an angle less than a right angle is called an acute (sharp) angle, and an angle greater than a right angle is called obtuse (blunt).

The degree and the use of the ordinary semicircular protractor may now be explained.

Notice that practice in measuring angles should be given

first, and *after that* practice in making angles to a given measure.

We are now in a position to start the real work of the term.

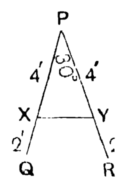
SCALE DRAWING. I

Have plans on which they can measure lengths and angles. Local maps, plans of class rooms, football fields, etc. Scales 1 in. to 1 mile, 1 in. to 1 ft., 1 in. to 10 ft.

If feasible, give them practical work: mark out places for drill on the gymnasium floor, mark out the football field.

After practice in measuring plans, they must learn to draw them.

A. Give them freehand sketches with dimensions marked.



E.g. a pair of steps--find the length of the cord. Make them copy the sketch freehand, state a scale, make the accurate drawing, state an answer.

After some practice at that,

B. Give them "wordy" questions,

(i) Make a freehand sketch* (not too small) and choose a scale. That must be passed by the teacher before (ii).

(ii) Draw the accurate figure and state the answer.

Much time will be saved by having (i) passed by the teacher before the boy goes on to (ii). A preparation might well consist of stage (i) for several questions, the next preparation being to do stage (ii) for the same questions; or (i) might be done in school and (ii) done for preparation.

Young teachers do not always realise that (i) is a difficult process for the boy: he has to translate from the words of a book into a freehand dimensional sketch. The teacher himself often dispenses with the freehand sketch, because he can visualise the figure; but the boy should not be allowed to omit the freehand sketch for some years.

* It is convenient to mark a right angle as in the figure:

The freehand sketch should always be shown up with the finished drawing, preferably on the same sheet.

It is important to aim at neatness and accuracy: in this first term of scale drawing it will need a lot of patience on the master's part. Some boys will be neat from the start, but many will have to develop a skill of hand without which their work cannot be neat.

Questions should be chosen with care and the teacher is advised to mark the questions in his own book, say "A" easy questions for the first few lessons, "B" harder questions to be set after they have had some practice, "C" questions only for the best boys.

In the course of this work the boy should have learnt the meaning of a bearing (e.g. N. 17° E.), angle of elevation, angle of depression, altitude of the sun. He will also have acquired the meanings of a lot of geometrical terms, and some skill in drawing and measuring.

Besides all this some boys will have guessed some geometrical facts, e.g. the angle-sum of a triangle, the equality of the base angles of an isosceles triangle. The master should certainly commend the boy for the guess, and he must use his discretion as to whether there is time for the boy to test his guess for other cases and perhaps to try to find a reason for the fact; but he should not let these discoveries distract the class from its main work.

CHAPTER III

A SECOND TERM'S GEOMETRY. ANGLES AT A POINT, PARALLELS, ANGLE-SUM OF A TRIANGLE

We now come to a new phase. We can improve our powers of scale drawing if we know some geometrical facts (theorems). This term's work will be concerned with the acquisition of some facts and their application to scale drawing.

ANGLES AT A POINT

The boy will now meet the formal enunciation of a theorem for the first time. How is the introduction to be made? In writing out the proof of a theorem the enunciation naturally comes first; but in introducing a boy to a theorem the enunciation must not come first.

The worst possible way of beginning a lesson would be to say "We will prove that 'If a straight line stands on another straight line, the sum of the two angles so formed is equal to two right angles.'"

(i) The words "prove" and "proof" should be reserved for their proper use, and we do not propose to give a formal proof of this theorem.

(ii) The class would not take in the meaning of the words used.

(iii) We want to develop initiative.

(iv) We must teach English: the class must learn to formulate their own enunciations, and gradually be led to perfect them under criticism from the teacher and from one another.

The fact that the sum of the two angles in this theorem is equal to two right angles can hardly be a surprise to boys who have used protractors; but the theorem is not one that calls for

verification by measurement. A very slight effort of thought by the boy will convince him that a general statement can be made; and when the general statement is so near the surface recourse to measurement is unnecessary, cumbrous and a loss of opportunity. In fact degrees should be kept out of the argument. The hand of a clock turns through two right angles between 12 and 6; if it does the journey in two stages the sum of the two stages is always two right angles. Or, "class—about turn" in two stages.

Now let them have a chance of discovering the fact for themselves. *Assisted discovery* must be the general rule when new steps are being made in mathematics. This is the way to make the subject interesting. But never forget that this is only half the battle; there must be drill as well; many young teachers forget or shirk the drill and their lessons are written on sand. One can make fun of the *Heuristic method*, and one can make it ridiculous by being too consistent and solemn about it; but as a teaching trick it is indispensable.

The discovered fact will be announced in imperfect form "Please, sir, they make up two right angles," and proceeding from this point we shall elicit that "they" are angles, and lastly angles formed in a particular way. This theorem is actually not very easy to phrase, but the perfect form should not be revealed till the class have done their best to express their meaning.

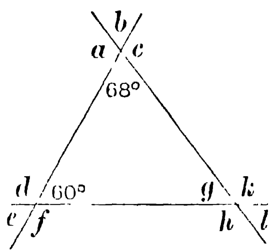
The boy will as yet feel no need for further formal demonstration. It is always difficult to get this theorem written out properly. The mere fact that boys find a thing difficult is not a sufficient reason for giving way; but if they are not ripe for the task, it is wiser to wait. To insist on formal proof at this stage would be wrong psychologically and so would dull interest.

But the enunciation should be learnt. Learning by rote is easy for children. To make them learn enunciations of theorems before they have won them would be wrong; to make them

memorise those they have won is a useful form of out-of-school work. This is a feature of the old teaching that should be retained. There have been complaints that the new teaching is formless and leaves no fixed impression. The learning of enunciations should help to correct this. A boy gains confidence if some part of what he has studied is fixed firmly in his verbal memory.

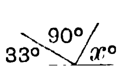
It is a good plan for each pupil to have a geometrical notebook in which to write the enunciations of all theorems that have been won.

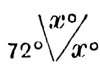
The result now needs to be driven home by numerical examples. The master can draw on the board various figures (e.g. a triangle with all its sides produced) giving numerical values to some angles; the class can then calculate some of the remaining angles in the figure.



It is simplest to label the unknown angles with a single letter as in the figure. At this stage angles g, h, k, l, cannot be found.

The term "supplement" and "supplementary angles" may well be introduced; the class may be asked for the supplement of 70°, 153°; $x°$ should also be considered for the sake of paving the way for algebra.





Other numerical examples are suggested by these figures; there is no need to have them in a book, the master can invent any number on the spur of the moment.

It is an easy step for the class to go on to "If any number of

straight lines meet at a point, the sum of all the angles made by lines taken consecutively is equal to four right angles."

Again, the enunciation must not be fired at the class; they must be led on to discover the fact first and then to enunciate it.

The master will easily invent numerical examples. He can also make the class see that this theorem follows logically from the previous one by producing* one of the lines.

By laying two pencils along one another and rotating one of them so that the letter X is formed the class may be led on to, "If two straight lines intersect, the vertically opposite angles are equal."

This may also be discovered by the class from logical considerations; and this raises the point

WHEN SHOULD LOGIC BE INTRODUCED?

The idea that young children can do things but cannot reason is responsible for much bad teaching.

I believe that actually children begin to use reason at quite an early age. A child in its cradle that has found that a cry will bring its mother uses that cry a second time—is not that a case of reasoning, even if it is unconscious reasoning?

It is important, I think, to distinguish in the case of young children between cases in which they see reason and cases in which they can express their reasons in words; many children will see a reason but will not be able to express it, none the less they are logical and we should respect that. But here is a nice instance of a child of $3\frac{3}{4}$ expressing a reason in words. The child had just learnt to count; that morning he had had a ride on a donkey and in the afternoon found his father "doing sums," and he wished to do sums. After various sums with pennies,

Father. "How many legs has a donkey got?"

Son. "Four."

* "Producing" will be a new word and will need explanation.

Father. "How many legs have two donkeys got?"

Son. "I must have some white pennies to do that." (There were only black pennies on the table.)

Son (after putting out four white pennies and four black pennies). "Two donkeys have eight legs."

Father. "How many feet have two donkeys got?"

Son. "Eight, of course."

Father. "Why?"

Son (after some thought). "Because they are joined on to the legs."

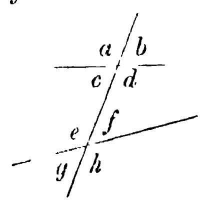
Most children would have been quite clear about the right answer, they would have had inside them the idea of a "one-to-one correspondence" but few children could have expressed their reason so simply.

I would say that at the stage of geometry which we are considering the child should use logic whenever he can, but he should not be a slave to it yet. Do not stop a child's progress for the sake of logic—our main business at present is to collect facts—but let those children who can see the logical connection between the facts use their power; with other children point out logical connections sometimes, but do not be worried if they seem not to respond; it is a great mistake to try to hurry the development of the logical power.

At this stage the boy must be given *vivâ voce* examples to apply the three facts he has now won. The teacher can supply plenty out of his head.

With a bright class I should certainly try little riders*. For example:

In the given figure $\angle b = \angle f$, prove that $\angle c = \angle f$.

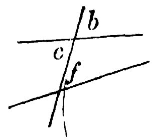


Let the boy draw his own figure marking only the angles with which he is concerned, and let him put it out formally.

* See chap. XI, and p. 288.

Data $\angle b = \angle f$.

To prove that $\angle c = \angle f$.



Proof $\angle b = \angle c$ (vert. opp.)

but $\angle b = \angle f$ (data),

$\therefore \angle c = \angle f$.

A great many riders can be obtained from this figure and the idea of a logical proof can be obtained from them. The rules of the game are soon appreciated.

We must not state that things are equal unless either

(i) they are given to be equal,

or (ii) we can justify the statement by means of one of the theorems we have already stated.

PARALLEL STRAIGHT LINES

To attempt to define parallel straight lines at this stage would be quite wrong. Children have the idea of parallel lines floating in their minds, it is the teacher's business to fix this. First ask for instances of parallel lines; it is then of interest for the teacher to try to find what the children regard as their essential feature, but the ideas that they have will not help as a rule.

In general, children regard parallel lines as everywhere equidistant; a little questioning will show the difficulty of finding their distance apart.

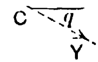
Sometimes a class may suggest that parallel lines never meet "How do you know that they never meet?" The class will soon see that there are difficulties here.

The teacher must then lead them to the idea that a cutting line cuts a lot of parallels all at the same angle. Draw a line across the rulings on a sheet of foolscap, mark the corresponding angles and ask the class for suggestions about the angles, perhaps let them measure them. Let the class draw parallel lines by sliding a set square along a straight edge*.

* I do not mean that they should learn and practice the method just now, but just draw one or two parallels in that way.

One of the best lessons I ever saw given to small boys was roughly as follows:

A man walks from A to B.



Master. "Where are his heels?"

Boy. "At B."

Master. "Where are his toes?"

Boy. "Pointing along BX."

(This was done by a boy coming up to the blackboard and pointing to the points and lines.)

Master. "He now turns on his heels so as to walk along BC. Where are his toes when he has turned?"

Boy. "Pointing along BC."

Master. "Through what angle has he turned?"

Etc., etc.

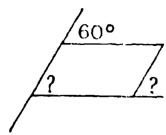
Master. "When he is walking along CD, he finds that he is walking in the same direction as at first. What can you say about the angles p and q?"

The lesson was made very vivid.

The class was then led on to state the two theorems:

"When a straight line cuts two other straight lines, if a pair of corresponding angles are equal, then the two straight lines are parallel."

Conversely. "When a straight line cuts two parallel straight lines, the corresponding angles are equal."



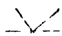
These results can be applied to numerical instances such as is suggested by the figure. The master can invent plenty of these; he must mind that the parallels go in various directions.

The boy has now got an important new idea, and I think that that should end a lesson; the idea wants to be digested (subconsciously) by the boy.

In the next lesson (preferably after an interval of a few days), refresh their ideas about parallels and corresponding angles;

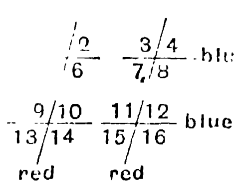
then deduce the corresponding theorems for alternate angles—this should be done logically (cf. the rider worked on p. 240).

These results must be applied to many figures. Here are a couple of important figures needed in the near future.



An excellent test at this stage is the following:

Draw two parallel blue lines and two parallel red lines and number all the angles in the figure. Write on the board pairs of numbers: let the boy write down whether the angles are corresponding or alternate, and whether the parallels concerned are blue or red.



The boy's paper will appear thus:

1, 9	Cor —B*.
10, 15	Alt.—R.

These can be easily marked (I have always given a mark for the "Cor." or "Alt." and another mark for "B" or "R") and the master can tell at once which members of the class have the idea; but I would warn him that he must test again with other figures (e.g. a parallelogram with a diagonal drawn), he will have to give a lot of practice before the whole class is perfect.

The next step in the theoretical work should be postponed for a week or two, and the gap can be filled up by going on to the next stage of scale drawing, but I will consider the rest of the term's theoretical work before considering that.

* For a boy who gets "Cor.—R" cover up the right-hand red line and ask what difference it would make if that were drawn in some other direction

The next theorems or facts to be taken are:

"When a straight line cuts two other straight lines, if a pair of interior angles on the same side of the cutting line are together equal to two right angles, then the two straight lines are parallel."

Conversely. "When a straight line cuts two parallel straight lines, the interior angles on the same side of the cutting line are together equal to two right angles."

These should be deduced from the theorems given on p. 241 (see the riders suggested on p. 239).

At this stage the method of drawing parallels by means of sliding a set square along a straight edge may be learnt. It needs some practice for a boy to acquire the necessary skill of hand.

The class may also be taught to draw a parallel at a given distance from a given straight line by the method suggested in the figure.

It is a good practical method, though not a strict "ruler and compass" construction.

The construction for drawing a perpendicular by sliding a set square may also be taken at this stage. But all these constructions may be postponed if thought desirable.

ANGLE-SUM OF A TRIANGLE

The class will probably have arrived at this theorem already. In these days of protractors, many boys discover it for themselves.

The Board of Education circular (No. 711) made the first

authoritative proposal that certain theorems should be postulated without proof, and this theorem was in the list proposed. But most people will agree now, I think, that it is best to prove it.

Let us apply three tests.

(i) Is the result so obvious that a class will regard a proof as tedious?

My own experience was that after doing Euclid I, Propositions 1–31, the fact that the angles of a triangle added up to two right angles came as a most glorious surprise. Not many boys get this glorious surprise in these days of protractors, but the result cannot be regarded as obvious.

(ii) Is the proof too difficult or elusive for this stage in a boy's development?

I think the answer must be "No." The proof is quite within their grasp, in fact most boys can be led on to invent it for themselves.

(iii) Is there any sequence difficulty?

In almost every logical system the proof of this theorem must depend on the angle properties of parallels.

So it seems that on all grounds the proof should be taken now.

How should the Proof be led up to?

It may be just worth while for each boy to draw a triangle and find the sum of its angles. This will suggest that the sum may be two right angles. I still remember my old mathematical master pointing out that such measurement could only suggest that the sum might be near to two right angles, and saying that, if we could measure the angles of a triangle whose vertices were in three distant stars, no doubt the angle-sum would differ from two right angles.

But this measurement is typical of much scientific discovery. Measurement and experiment in a limited number of cases may suggest a possible general truth; the usual course is for the experimenter to test this supposed truth in other cases—

remember that one single case which gives a result not in agreement disproves the supposed truth, provided the work is accurate. The experimenter then tries to get a theoretical proof.

Another nice experiment is to cut out a large paper triangle, tear off its corners and fit them together and so make the angle-sum.

Having done all this, the class can be taken on to the ordinary proof suggested by the second figure on p. 242. They will get it for themselves if they are given the figure and told to mark angles that are equal.

The proof by drawing a parallel to the base right through the vertex should also be considered; but the other proof is to be preferred, because it also proves that "the exterior angle formed by producing a side is equal to the sum of the two interior opposite angles."

At any time now the class may be introduced to the various names applied to triangles: right-angled, acute-angled, isosceles, etc.

Naturally the class will go on to the angle-sum of a quadrilateral, and will deduce it by drawing a diagonal.

For the polygon it seems to me that the sum of the exterior angles formed by producing the sides in order is much more striking than the property about the sum of the interior angles.

The property may be discovered by considering the angle turned through at each corner when walking round the polygon.

A certain amount of *vivâ voce* work may now be done about the angles of regular polygons.

Throughout all the first two terms' work let the boy be on the look-out for interesting facts. The teacher should not be led far from his path by them; but it is worth encouraging the boy to try to discover things and, if the discovery can be followed up at the time, the interest of the class will be much increased.

SCALE DRAWING. II*

In Scale Drawing I, we used no theorems. Now we have a fair stock of theorems to help us. First of all lay down a set of rules.

(i) A freehand sketch must be made and shown up with the finished drawing.

It should be neat and not too small.

Point out that later on, especially when we have trigonometry at our command, the freehand sketch is all that is necessary. Even now we may get a question which it is possible to answer from the freehand sketch without drawing an accurate figure.

(ii) Mark the data in the sketch†.

(iii) Mark in the sketch other angles and lengths which can be found by aid of the theorems, e.g. alternate angles from parallels, or vertically opposite angles.

(iv) Choose a scale and state it. *Vivâ voce* practice should be given in this.

(v) Make the accurate drawing.

(vi) Do not rub out any construction lines.

(vii) State the answer in words.

Note that, where the method of construction is not obvious from the figure, it should be stated.

The question will arise, what units should be shown on the scale drawing: the length represented or the actual length of the scale drawing? I think the answer is to be found from an architect's or engineer's drawings; they always show the length represented, but there is no objection to showing both.

Again, teachers should be very careful in choosing the questions set and should mark questions in their own copy of the text-book "A," "B" or "C" as suggested on p. 234.

* Compasses not used here.

† It is convenient to distinguish in the freehand sketch between the data and deductions from the data. The data may be put in in ink (perhaps dangerous) and the rest in pencil; or the data may be underlined.

By the end of this second term's work the class should have acquired (i) a good deal of skill in scale drawing, (ii) a fair store of geometrical facts (theorems), and (iii) the idea of a logical proof.

With a skilful teacher, the class should be full of interest and enthusiasm and ready for more.

Contrast this with the old teaching of Euclid.

The late Canon J. M. Wilson once told me that to write $\angle BCA$ for $\angle ACB$ would have been counted as a mistake in his young days.

The late Master of Jesus College, Cambridge, used to tell a story of an undergraduate who thought he must have passed in Euclid; "I had learnt ten propositions by heart and got eight of them in the examination, I got them right to a comma—I do not know whether I put the right letters at the right corners, but I do not suppose that that matters."

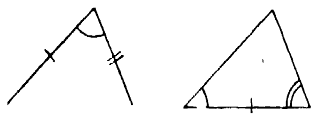
CHAPTER IV

A THIRD TERM'S GEOMETRY. CONSTRUCTION OF TRIANGLES, CONGRUENT TRIANGLES

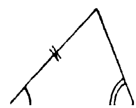
Again in this term our business is to acquire more facts, or theorems, to help us in our scale drawing.

Our first business is to consider the various cases of construction of triangles from given data.

By considering the two cases indicated in these figures, the pupil jumps to the conclusion that, if he is given three suitable parts of a triangle, the triangle can be drawn and is fixed.



He will naturally like to consider other possible sets of data. He will see that the data suggested in this figure is equivalent to that of the second figure above; for by aid of the angle-sum property, the third angle in the figure can be found.

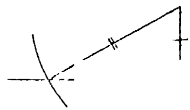


The case of the three sides being given is rather different, and will introduce a new idea. The class will see that three rods of given length can only be made into a triangle in one way, but the problem of constructing a triangle whose three sides are given is not so easy. Suppose that the three sides are to be of lengths 9 cm., 8 cm., 7 cm.; first of all draw QR = 9 cm. Now P is to be 8 cm. from Q, mark a lot of points 8 cm. from Q; how could we mark *all* the points 8 cm. from Q? This introduces the use of a pair of compasses. We can draw the circle of radius 8 cm and centre Q, and also the circle of radius 7 cm. and centre R; P has to lie on both these circles. Hence P is determined.

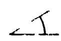
It would be tedious to the reader if I laboured the point here, but it must not be hurried with the class.

Some bright child is sure to suggest that the construction gives two triangles, but it is easy to convince the class that the triangles are congruent (to use the ultimate word).

It remains to show that a right angle and the lengths of the hypotenuse and one other side fix a triangle*. The construction will often worry a boy, but he will soon discover for himself that he must draw the right angle first.



It is also well to look briefly at the so-called "ambiguous case*," two sides and a non-included angle are generally insufficient to fix the triangle. Draw the angle first and then the adjacent side, use the compasses to get the third vertex, but they generally give us two possible positions for the third vertex.



The class should now be clear (i) that three given parts in general fix a triangle, (ii) what sets of three parts fix the triangle uniquely.

The class may now have some practice in drawing triangles from given data†; this is perhaps the place to teach them to estimate hundredths of an inch and of a centimetre.

Naturally this leads on to the construction of quadrilaterals and the consideration of the number of parts necessary to fix a quadrilateral, though there is no object in labouring the latter point; but it may be interesting to digress if time permits and see that four sides do not determine a quadrilateral, but produce

* Both these will naturally be approached by giving the class data (for constructing a triangle) that leads to these cases.

† It is a good thing sometimes to let them choose their own data; they will probably then find out that (i) two sides of a triangle must be greater than the third side, (ii) two angles of a triangle must be less than two right angles, (iii) three angles do not determine a triangle. It does not matter if these are not found out, but the class will be interested if they do.

a deformable frame—this may lead to the consideration of frameworks and the nature of the additional constraints needed to stiffen a frame.

A third section of scale drawing, involving the use of compasses, may be taken at any time now. And henceforward scale drawing as a goal slips into the background; the class should have been educated by now to have the higher aim of acquiring geometrical knowledge, a knowledge of theorems and (to some extent) their dependence on one another.

CONGRUENT TRIANGLES

Before considering the modern treatment of this subject, let us recall the conditions that existed in the opening days of this century. Think of the state of mind of a boy of twelve set down to learn Euclid I, 4 (the two sides and included angle case of congruence). At the worst he saw two obviously equal triangles and was required to prove them equal. The first advance in this crude presentation would be to make him draw two equal angles and two equal pairs of including sides, leaving the base open. This enables him to grasp that there is something to prove; but only a very docile boy, or a born mathematician, would really believe that such an obvious fact is worth proving at such length. Thus there arose that rebellious and scoffing attitude of mind familiar to teachers of geometry in the last century.

And all the time the boy was right and the teacher wrong. The superposition proof is no proof. A triangular piece of paper can be moved and superimposed on another piece of paper; but we profess to be moving a geometrical figure. A figure consists of points and lines; a point has position and nothing else motion means change of position, a different point. How can a point be moved? The demonstration therefore referred to material objects and not to geometrical figures.

But even if this is admitted, our case was not much better. How do we know that the sides of the triangle do not alter *en route*?

Of course, if doubts are entertained about this it is possible to move the triangle back to its original position and verify that it has not changed. All that this proves is that when restored to its original position it is unchanged. But what were its dimensions in mid-journey?

All this is very difficult, but the long and short of it is that Hilbert finds it necessary to *postulate* that the bases are equal.

Now all these criticisms on proof by superposition would be beside the point if the proof really convinced the boy. But the average boy did not believe that there was anything to prove. A proof that is neither sound nor convincing is almost worthless.

The modern way of viewing the matter should be briefly as follows:

Three suitable data are enough to specify a triangle in everything but position. Triangles drawn to the same sufficient specification in different positions are equal in all respects. It does not matter whether we make our construction in Harrow or in London. The only question for the boy is "What data are necessary to determine a triangle in shape and size?"

This is a comprehensible question, worth answering. We have already seen that the triangle is fixed if we are given

(i) two sides and the included angle,

(ii) two angles and one side (it must be specified whether the side joins the angles, or is opposite to one of them and in that case to which it is opposite),

(iii) three sides,

(iv) a right angle, the hypotenuse and one other side.

The class will have actually constructed triangles from sufficient data supplied, but, at this stage, it is not a matter of accurate drawing. A conscientious teacher might provide each boy with a piece of tracing paper, make him draw a triangle

with $AB = 4$ in., $AC = 5$ in., $\angle A = 32°$, and finally fit his triangle on to his neighbour's. It fits—or does not fit. The boy sees at once that it is simply a question as to whether the drawings are accurate. It does not strengthen his conviction that all the triangles *must* be equal—this is intuitive. The effort of drawing will distract his imagination.

On the other hand, if he is told to *imagine* the triangle drawn in such and such a way, and urged to see that the triangle is completely determined, his imagination may not function. He may be gazing in an interested and attentive way at his master and thinking of something quite different. The imagination will not be bidden: it must be stimulated by something visible and preferably tangible.

Avoiding the two extremes just mentioned, the teacher will no doubt find various possible ways of presenting the matter And let him seek for contrast. Compare the various cases of three sufficient data with cases of three insufficient data (three angles, or two sides and a non-included angle).

When the class has stated the various cases of congruent triangles, they must learn to apply them to straight-forward riders*. It is not easy to find many simple riders that are not obvious by symmetry, but the various theorems about the parallelogram provide some. A well-taught class can be led to take an interest even in cases that seem obvious by symmetry, and those cases bring out the meaning of what a proof really is, it does not depend on whether the result is obvious, but only on the data and previously established facts.

THE ISOSCELES TRIANGLE. HYPOTHETICAL CONSTRUCTION

It is natural to bring in here the two theorems about the isosceles triangle. They will have been discovered by an appeal

* See chap. XI, p. 288.

to symmetry, but both theorems are easy to prove by congruent triangles if the bisector of the vertical angle is drawn. Euclid avoided this method because he had not shown how to draw the bisector. This is the famous question of **hypothetical construction**. He never uses a construction that he has not performed and proved. But it is clear that a bisector exists; whether or no we can draw it with ruler and compasses is irrelevant; the bisector is there all the time, though we may not have put a pencil mark over it: we can close our eyes and imagine the figure if necessary. The real basis of the objection is this: unless we have discovered how to draw a required line in a figure, it is quite possible that such a line may not exist. For instance, the attempted proof of a theorem might depend on drawing a circle through four points; and generally such a circle would not exist; if I am compelled to draw the circle before I use it, I shall certainly discover the impossibility of the suggested construction. So that if the rule is relaxed and I am permitted to use lines that I have not constructed, I must first make sure that the lines exist. In the case of the bisector of an angle, the existence is obvious. Euclid's restriction is artificial and inconvenient; for example, he is debarred from proving any properties involving the trisection of an angle.

A THEOREM AND ITS CONVERSE

Here we must draw attention to what is meant by a converse theorem. The statement of a theorem can generally be divided into two parts (i) the data, or hypothesis, (ii) the conclusion.

If the hypothesis and conclusion are interchanged, a second theorem is obtained, which is called the converse of the first theorem.

Thus Theorem 12 states that, if $AB = AC$, then $\angle C = \angle B$; the converse, Theorem 13, states that, if $\angle C = \angle B$, then $AB = AC$.

Now ask the class for instances of theorems and their converses which they have met; state theorems and ask them to state the converses.

But the class must be warned that the converse of a true theorem is not necessarily true; for example,

"In a triangle, if one angle is a right angle, two of its angles must be acute."

The converse theorem is that "In a triangle, if two of its angles are acute, one of its angles must be a right angle," but this is untrue.

If the class keep notebooks in which they make lists of all theorems which they have won, it is useful for them to mark with a * those theorems which have true converses.

CHAPTER V

REVIEW OF THE FIRST YEAR'S WORK

We have now considered a year's work. The class should have acquired considerable skill in scale drawing, but the emphasis should have been gradually shifted from that to the acquisition of theorems and logical work. The fundamental theorems have been made clear and should be available for future use: in fact, the field of theoretical geometry is open before us. Still in the near future we have to act as pioneers in an unknown country; we shall not yet attempt to build up logically all that we learn, but we shall have at our command for exploring not only scale drawing but also the fundamental theorems of geometry and a certain amount of logical power. But, before we go on to explore the new country, there are a few questions we will discuss now; and, in the next chapter, we will consider various teaching questions.

DEFINITIONS

In the early stages a boy should not be bothered with formal definitions; he wants to get a working knowledge of the language of geometry; he must understand the meaning of words and expressions rather than be able to define them in set terms. For example, he wants to acquire a knowledge of the various properties of a parallelogram without bothering at first which is the fundamental property (taken as the definition) from which the other properties can be built up by logical argument.

We may divide definitions into several classes:

(i) Philosophical definitions—e.g. point, straight line.

The boy and girl are not really concerned with these; all that is necessary is that they should have a clear conception of what

the thing is—a formal definition does not help them, and does not concern them.

(ii) Explanatory definitions—e.g. acute angles, alternate angles, complementary angles, diameter of a circle. It is interesting to divide the adjectives used into

(*a*) Adjectives of quality or position—e.g. acute (angles) alternate (angles), vertically opposite (angles).

(*b*) Adjectives of quantity—e.g. complementary (angles), equilateral.

(iii) Definitions which are essential for logical purposes, e.g. isosceles triangle, parallelogram, square.

These definitions of class (iii) should be learnt by heart. It is quite true that a parallelogram has its opposite sides equal and parallel and that its opposite angles are equal; out of the various facts it is possible to choose several different sets which might be taken to define a parallelogram; but for a logical system it is essential to agree on one standard set as fundamental, and the pupil must remember what this standard set is.

GEOMETRICAL VOCABULARY

It is essential that geometrical words should be used in their correct sense; though it is sometimes useful to use a popular phrase or a roundabout phrase at first (e.g. "equal in all respects" for "congruent"), the use of the technical phrase should not be postponed too long; it may be well to use both phrases for a time (e.g. "two congruent triangles, i.e. two triangles that are equal in all respects"), but as soon as the word is thoroughly understood and familiar it is an economy of time and thought to use the technical phrase. But it is essential that the technical phrases should not be used unless they are really understood I was very surprised in my young teaching days at finding boys who did not distinguish between "a rectangle" and "a right angle"; such glaring instances do not occur today, I hope, but I still hear of boys who muddle "rectangular" and "recti-

linear," and many boys try to cover up ignorance by using technical phrases which they do not understand.

To avoid boys missing some technical phrase it is useful sometimes to run through the index of a geometry book and see whether any have been missed by any of the class.

I find that the word "hypotenuse" is not always clear, and the phrase "angle subtended by" is very frequently misunderstood.

CHAPTER VI

MISCELLANEOUS TEACHING POINTS

ATTENTION

In *vivâ voce* work avoid the use of letters that sound alike, e.g. B, D, E. If it is a strain to a boy to decide which letter was said, his attention must flag. Some people would say that it is good to encourage clear articulation; I agree, but that is not the point; we do not want to use letters that cause boys' attention to flag either through having to think of clear articulation when geometry is the job in hand, or through uncertainty.

Nothing is more tiring to masters and boys than following a geometrical argument in a complicated figure,

$$\angle AHF = \angle ABD, \quad \angle DHC = \angle FBH.$$

I call this "letter-chasing." Boys' attention must flag where work on a blackboard involves letter-chasing; I would advocate letting boys refer to the "top angle at B," "the angle on the left" and so on. Coloured chalks are very helpful in this connection, "the red angle" is easier to pick up than the "angle XPF." Of course when a boy is writing out a theorem or a rider he should certainly write "$\angle XPF$"; he must learn to do the formal thing and it is shorter to write. Of course, if there is only one angle at a point A, he may write "$\angle A$"; but he must be warned against writing "$\angle A$" when there is more than one angle at A.

In the middle of a *vivâ voce* lesson it is a great relief to the class to have to write for a few minutes; they find it easier to attend after this short relief.

THE USE OF SCRAP PAPER

Suppose we are going to do on the blackboard several riders from a book. Let every boy have a sheet of scrap paper before

him and let him draw his own figure before the master draws it; this gives him valuable practice in interpreting the words of the book. After the master has drawn his own figure on the board, it may be advisable for the boy to re-draw his figure so that it agrees with the master's figure; and in any case he should letter his figure in the same way as the master has. Then the boy should be given a few minutes to mark in his figure the various facts he notices and to try to solve the rider. After that the master may get the class to help him to solve the rider.

A master who has not tried this may at first feel that time is wasted, but I am sure that he will soon come to the conclusion that the class have learnt more than they would have done if they had not drawn their own figures and made their own attempt at the rider first. Very soon too the master will find that the class get so used to the procedure that little or no extra time is needed.

This same use of scrap paper may be adopted in revising theorems. And further, perhaps the class are agreed that two certain triangles have to be proved congruent, each boy should write on his scrap paper the three facts that prove them congruent. This gets much more out of the class than merely asking one individual to give the three facts or asking three individuals each to give one fact; it keeps the class more attentive, and, if a boy has made a mistake, it is much better to deal with it at once.

STYLE IN WRITING OUT RIDERS AND THEOREMS

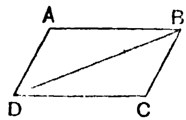
I think that, when a boy writes out a rider or theorem formally, he should certainly give his data and state what he wants to prove. Part of the difficulty which boys have in doing riders lies in the fact that they are not clear as to exactly what is given; the data should be as explicit as possible and in the form which is most convenient for use, e.g., "ABC is a triangle, M is

the mid-point of BC" is not so good to my mind as "ABC is a triangle, BM = MC."

I acknowledge that in the second form to be absolutely explicit it should also be stated that M is in BC, but I think the boy may be allowed to assume that from the figure. I should not mind very much if he merely wrote "BM = MC" and left the master to assume from the figure that ABC was a triangle.

Again, "ABCD is a quadrilateral with one pair of opposite sides equal and parallel" is indefinite and not so helpful as "ABCD is a quadrilateral with AB equal and parallel to DC." I would even forgive the boy if he merely said "AB is equal and parallel to DC"—or, "AB = DC and AB is to DC."

Of course reasons must be given clearly.



I find boys often say (see the figure)

$\angle ABD = \angle BDC$ (alt. $\angle$'s).

I think he should write,

$\angle ABD =$ alt. $\angle BDC$ ($AB \parallel$ to DC),

and I should not mind much if he here left out the "alt."; but I always tell boys that "alternate" is an adjective of quality (position), not of quantity, and the reason the angles are equal is because the lines are parallel; alternate angles are only equal when lines are parallel.

THE USE OF TECHNICAL TERMS AND ABBREVIATIONS

As soon as a boy fully appreciates a technical term, he may be allowed to use it; but the use of technical terms is sometimes used to cover ignorance and the teacher must constantly be testing whether boys really understand the terms they use.

In the same way boys should be allowed to use abbreviations from the start if they really understand them. It is shorter to write "$\angle ABC$" than "angle ABC" and "alt. $\angle ABC$" is clear enough for "alternate angle ABC." I should say, let the boy use abbreviations as soon as he feels the need for them.

The use of abbreviations should be limited to their legitimate use, e.g., abbreviations should not be used in the enunciation of a theorem and "Δ's ABC, DEF are $\equiv$" should not be allowed.

NEATNESS AND DISTINCT WRITING

A master should insist on neatness and good writing from the first. The letters of a figure and references to them in the written work should be in block capitals. How often a boy writes D so that it looks like O!

FREEHAND FIGURES

The boy should draw his figure for a theorem or rider freehand; if necessary he may draw his circles with a pair of compasses, but he should train his hand to draw well enough freehand. (There is such a thing as waste of time through excessive craving for neatness.) A freehand figure should be neat, reasonably large and the letters in it should be very clearly printed.

REFERENCES FOR CONGRUENT TRIANGLES

I believe the following way of referring to the various congruence theorems is due to Mr E. A. Price of Osborne and Dartmouth. I do not think it is generally recognised and I do not recommend that it should be, but it is useful for private use for a boy to show that he knows which cases of congruence he is using.

S.A.S.—two sides and the included angle.

A.S.A.—two angles and the side joining them.

A.A.S.—two angles and the side opposite to one of them.

S.S.S.—three sides.

S.S.A.—would be two sides and a non-included angle, and so would not be true (perhaps A.S.S. would be better).

R.H.S.—right angle, hypotenuse and a side.

NOTEBOOKS

I have already suggested that each boy should keep a notebook in which he writes the enunciation of all the theorems which he has done, and indicates which of them have converses that are true. He should also write in it the few definitions which he has to memorise.

USEFUL REVISION

Occasionally a class should be told to write out the enunciations of all the theorems which they can remember. An experienced master tells me that he marks such work by giving one mark for each enunciation which more than one boy gives, but three marks for an enunciation which only one boy gives.

KEEP A RECORD OF WHAT EACH BOY KNOWS

If a master is teaching the same part of a subject to two different classes, he is very apt to forget what he has done with each class; some men have taught a subject so often that they may think at times that they have done a particular piece of work which they have not done with that class.

To meet this difficulty, I would strongly advise a master at the beginning of term to write out in his mark book, or a notebook, a fairly detailed syllabus of the work he proposes to do. then he should tick off each piece of work as he does it, and perhaps a different tick when he has revised it.

Again, a master doing a first year's geometry might well have columns in his mark book headed with headings such as I am suggesting below, and tick off each boy when he seems satisfactory on the subject concerned.

Here are a possible set of headings for the first term:

(i) Measure in inches and tenths, (ii) in centimetres and millimetres, (iii) idea of an angle, (iv) bearing, (v) angle of elevation,

(vi) angle of depression, (vii) altitude of sun, (viii) neatness, (ix) accuracy, (x) stating an answer in words.

Experience will enable a master to add to or delete from this list, as he feels necessary. This list will also help him to deal with boys who have missed some lessons.

For the second term the list might be as follows; but it is easier to use small figures than the words I have given below:

(i) St. line standing on another st. line, (ii) vertically opposite angles, (iii) corresponding angles, (iv) alternate angles, (v) interior angles, (vi) the last figure on p 242, (vii) angle sum of triangle, (viii) exterior angle of triangle, (ix) sum of exterior angles of a polygon, (x) notes as to capacity for doing riders.

For the third term, the headings would be the various cases of construction of triangles, the various cases of congruent triangles, the isosceles triangle, and notes about capacity for doing riders.

MARKING TEXT-BOOK

I have already suggested that the master should mark in his text-book (i) the easy questions to be done in the early lessons, (ii) those to be done later, (iii) the questions that can be used to keep the better boys quiet

CONTINUOUS CHANGE IN A FIGURE

It is very instructive to get a class to think of the changes that take place in a figure as some element in the figure is changed. References to this are made on pp. 268, 275. Here are a few additional suggestions that might be used when suitable occasions arise.

The sides AB, AC of a triangle ABC are of constant lengths; trace the changes in BC as $\angle$BAC increases from 0° to 180°.

The base of a triangle is fixed and the height is constant; how may the vertex move? Trace the consequent changes in (i) the vertical angle, (ii) the remaining sides.

The base of a triangle is fixed and the vertex is moved about anywhere. What happens to the line joining the mid-points of the moving sides?

AB and $\angle$BAC (acute) are fixed dimensions of a triangle ABC, and the side AC increases continuously from zero; trace the changes in the length of BC.

CHAPTER VII

THE SECOND YEAR'S GEOMETRY

Boys should not be rushed on to this until they are sound on the previous work. At the end of a year most boys will probably be ready to go on to what I have described as the second year's geometry; but some boys below the average will need at least another term at the earlier work.

On the other hand, if a boy is kept grinding term after term at the earlier work, he will be bored; many boys' interest in geometry is not really aroused till they can get on to the circle.

In the case of preparatory schools, if a boy is nearing the time that he will be going on to a public school, it may be wise that he should only do the work referred to in chap. VIII; from the public school point of view it is far better for him to have done that work thoroughly than that he should have been taken through the work of chaps. VIII, IX and X, and be unsound on it all. I do not feel that the same applies to a secondary school, where he can be put through all three chapters again.

In the following chapters I have dealt with the work in the order in which it seems usual to take the subjects. But it is perfectly easy to take the work discussed in any one of chaps. VIII, IX and X before that in any other of the three chapters

Many schools are bothered with removes every term and a master is faced with a class half of whom have done the work of chap. VIII, and the other half have only done the first year's course. This difficulty would be met and has been successfully met by the following plan.

In the autumn term two or three consecutive sets or divisions all do the work of chap. VIII, in the spring term all do that of chap. IX and in the summer term that of chap. X.

This has the advantage that each of the sets can work as one unit, and in whatever term a boy joins either of those sets he will in the course of a year have covered the whole ground; in case he still stays in that group of sets for a fourth or even a fifth term, the work in those terms will not be new to him, but it will be a year since he did it so that it will have some freshness and he will never be doomed by the exigencies of removes (which have often to be settled through pressure of numbers) to do the same work in two consecutive terms.

There is just one difficulty that arises in the case of a boy who does the work of chap. X before that of chap. IX: he will want the Theorem of Pythagoras for various calculations in connection with the circle, but it is quite easy to give him that.

Some people will feel that chap. VIII must be taken before chaps. IX and X, and there is something to be said for this. If they adopt this course, they might apply the suggestion I have made above in a modified form: they might take chaps. IX and X in alternate terms.

If either of these plans is followed, the average boy who makes normal progress will have covered the ground to the end of the circle in two years from the time at which he started geometry seriously. If he is at a preparatory school and started geometry at 11, he will have covered this ground by 13 and he will have time for revision before his entrance examination to a public school, or, if he is really good, he can go on to other work.

CHAPTER VIII

THE PARALLELOGRAM, THE MID-POINT THEOREMS, RULER AND COMPASS CONSTRUCTIONS, LOCI

THE PARALLELOGRAM THEOREMS

The parallelogram theorems are pretty sure to have been done already as riders, but now they must be added to the boy's stock in trade.

First of all it must be impressed on the class that, though a parallelogram could be defined in various ways and its properties derived from the definition chosen, for a logical system it is essential to adopt one standard definition, and the natural one is "A parallelogram is a quadrilateral with its opposite sides parallel."

Having agreed on this definition, the class will easily prove the various parts of the direct theorem. The converses will present no difficulty except "A quadrilateral is a parallelogram if both pairs of opposite angles are equal." Sometimes a class is worried as to what the converses really are, or why they should be proved; this difficulty can be met if the teacher expresses a converse in this form "If I construct a quadrilateral so that its opposite sides are equal (but take no other precaution), prove that the opposite sides will of necessity come parallel."

There is just one point about the proofs of the converses. Should the proofs be made to depend on one another like a string of sausages, or should they all go back to the definition and hang like a bunch of bananas? E.g., in proving that "the figure is a parallelogram if one pair of opposite sides are equal and parallel," should the boy be allowed to assume that "the

figure is a parallelogram if both pairs of opposite sides are equal." The answer is emphatically "No." In each case he should go right back to the definition and prove that the opposite sides are parallel, he then avoids any bother as to the order in which the converses come; there is no need to remember the order, and in an examination there is no question of the examinee's order differing from that of the examiner.

The teacher will naturally go on to discuss the special cases of parallelograms, the rhombus, the rectangle, the square. He may also develop the idea of a changing figure; he might have a parallelogram made of four rods hinged at the corners (I have one made by a boy with rods of 30 and 20 in., the hinges are rivets put through the overlapping rods). As the parallelogram is deformed, notice the changes in the length of each diagonal: are they ever equal to one another? What special shape can the figure take? What happens to its area* as it moves? When is the area greatest? Notice the changes in the angles between the diagonals; can they ever be right angles? Consider the special case of a rhombus.

THE MID-POINT THEOREMS

The class will easily prove these after a little struggle with the construction (see chap. XI, p. 292).

Here the class may feel that, whichever of the two theorems is taken first, the uniqueness of the line across the triangle makes the converse obvious without proof. I would encourage them to have such ideas and show how reasonable their ideas are; but I would point out that, if they give an independent proof, then they are not dependent on their memories to know which theorem comes first, in fact some books take one theorem first and some the other.

The riders "If the mid-points of the adjacent sides of a

* There is no reason why its area should not be considered even at this stage.

quadrilateral are joined, the figure thus formed is a parallelogram" and "The straight lines joining the mid-points of opposite sides of a quadrilateral bisect one another" are surprising and give the teacher a pleasant opportunity of going into three dimensions.

RULER AND COMPASS CONSTRUCTIONS

The first thing perhaps is to point out that Euclid laid down that the only instruments to be used were a straight edge (not graduated) and a pair of compasses; when we speak of "ruler and compass constructions," we limit ourselves in that way. Again, in general, if we are asked to *draw* a figure we may use any instruments; if we are asked to *construct* it, we may only use ruler and compass*.

It is well to make a class realise that, with reasonably good instruments, most of these constructions are more accurate than constructions in which measurement by means of a graduated ruler or protractor are made; in particular, in constructing a right angle we do not depend on the accuracy of a set square. On the other hand, parallels are drawn more accurately by sliding a set square along a straight edge, and here we do not depend on the accuracy of the right angle of the set square.

In acquiring these constructions, the class will have an appeal to their sense of symmetry and this will give an opportunity for a digression on symmetry. They should know all the usual constructions and the proofs of these will provide useful rider work.

It should be pointed out that the construction for bisecting a straight line is made by drawing the perpendicular bisector of the line. How often does a teacher find a boy writing "Bisect **AB** at **C**, at **C** draw **CD** perpendicular to **AB**" instead

* This distinction is not universally recognised yet, though the Mathematical Association has asked examining bodies to adopt it; but it is some guide to the boy if used with discretion.

of "Draw CD the perpendicular bisector of AB cutting AB at C"!

Once again the class should be reminded not to rub out any of the construction lines.

SYMMETRY

There can be no doubt that a boy has some innate sense of symmetry. Ask a class of beginners to draw a triangle, the majority will draw a triangle which is approximately isosceles, ask for a four-sided figure, most of them will draw a figure that is approximately symmetrical, some will even draw a square.

Geometry teaching in the past, instead of using and developing this sense of symmetry, has ignored it or discouraged its use; Euclid had no propositions on it, nor have most modern books

I have seen attempts made to make symmetry fundamental in a theoretical course and so to shorten the time spent on congruent triangles; with some teachers it has proved very successful, but with the majority (whom I grant were not enthusiastic about trying it) the results were thoroughly bad; neither the boys nor the masters seemed to appreciate how it could be used theoretically. The results of these experiments lead me to the conclusion that there is great difficulty in it for any but exceptional teachers, and that it is unwise to make symmetry one of the fundamentals in a theoretical course.

But it is certainly desirable that the boy's innate sense of symmetry should be developed; the boy will certainly find it interesting and it will ultimately give him some sense of power, or at least it will teach him that there is some rhyme and reason in his innate sense.

He should be encouraged to see that in many figures the one half may be folded about a line so that it fits on the other, and so to deduce the equality of pairs of elements.

A very good way of developing the idea is to give a boy a

figure and the axis of symmetry, and make him draw the rest of the figure.

The teaching should aim at leading up to the following two main facts: in a figure that is symmetrical about an axis,

(i) the line joining corresponding points is bisected at right angles by the axis;

(ii) a line and its image intersect on the axis and are equally inclined to the axis.

With these at his command he has a new power of attack on new work, though he will resort to the longer method of congruent triangles for his ultimate theoretical proofs.

Symmetry about a centre is more difficult and the boy has not got the same innate feeling about it; but it is well worth developing with some boys if time permits.

LOCI

Experience in teaching and examining boys from many different schools leads to the conclusion that the idea of a locus is not as widely understood as it should be. An examiner of great experience in School Certificate examinations wrote to me some years ago "Why does the average examinee say 'the locus of a point equidistant from two points is *on* the straight line, etc.'? The 'on' is too frequent to arise from carelessness." Boys do not have enough practice in using the word and the idea. It must be the case that many boys learn the two standard theorems and do little else. But the idea is much too important to be dropped so soon. There are at least two good reasons for dwelling on it.

In the first place, the thing is interesting. It leads easily beyond the straight line and circle. With very little trouble the boy can plot loci which are parabola, ellipse, hyperbola, limaçon, cardioid, cycloid and other curves. I have known boys of quite ordinary ability who voluntarily spent hours of their free time in plotting the locus of a point on a sliding window bar, etc.

The idea appealed to their imagination. If a teacher can once do this, the battle is won.

A second reason for dwelling on loci is that in all problems of construction the required point or points must be found by the intersection of loci. Unless the locus idea has been well digested, constructions will be performed by the illegitimate spotting method.

Therefore let loci be studied with deliberation, even at a first reading. First let the class get used to the word and the idea Ask questions such as this:

What is the locus of a man's hand as he winds the starting handle of a motor car?

It is easy to make a large number of simple exercises in which the nature of the locus can be discovered by intuition and without lengthy graphical work. Loci in three dimensions should be considered. And when the two standard theorems are mastered, there is a good variety of problems on intersection of loci.

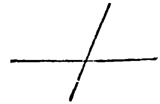
The work leading up to the two standard theorems can be made very interesting. Let each boy mark two points, A and B, on his paper, then mark a number of points at equal distances from A and B—do not use the word equidistant too soon. what pattern do these points form? The boy's intuition and sense of symmetry should lead him to see what the locus is.

Here we may bring in the common stages in geometrical discovery, (i) the guess, (ii) the verification of it by measurement, (iii) the logical proof.

LOCATION OF A POINT

The idea of using loci may be driven home by considering the various ways in which a man hiding a treasure, say on Dartmoor, could fix its position.

On Dartmoor there are two main roads running roughly as in the figure and there are many conspicuous tors.



He may fix a position as:

(i) in line with two points (e.g. in the line between two tors),

(ii) at a given distance from a point (e.g. a tor or a cross-road),

(iii) at a given distance from a line (e.g. a road),

(iv) equidistant from two points,

(v) equidistant from two lines.

If we are given any one of the pieces of information (i)–(v), we are given a locus on which the treasure must lie; if we are given any two of (i)–(v), we have two loci on each of which the treasure must lie; hence, if we dig at the point, or points, of intersection of these loci, we must find the treasure.

Working on the blackboard, the master may draw one locus in red and the other in blue; the required point must be both red and blue.

In the early stages the loci given by (i)–(v) are all that we want to use.

CHAPTER IX

AREAS AND PYTHAGORAS

AREA

Area is perhaps a primitive idea; in the presence of two slabs of chocolate of the same thickness, no doubt the youthful mind would make a rough estimate of their areas. The class will have already done something about area in arithmetic, but this will be a good opportunity of consolidating and clarifying the ideas they have got.

How do we measure area? Let us go back and consider how we measure length. Two boys have tried to roll a cricket ball so as to stop on a certain line; they can see which is nearer by one of them measuring with his feet. That is by taking some particular length as unit, and seeing how many times that unit is contained in each length. This will give the teacher an opportunity of digressing on the history of the standard foot.

In the same way, if we want to compare two areas, say the area of two blackboards, we can take some standard area (I always take a duster) and see how many times it can be fitted on each board. At once we see that the method is inconvenient, though it is fundamental.

We naturally go on to define the square foot and square inch, divide the boards up into square feet and count the squares. Then we go on to show that we can economise in our counting, and so lead up to the rule for finding the area of a rectangle. This will all have been done before in arithmetic (see "Arithmetic," chap. VII), but it is valuable revision here and the fundamental idea will in many cases have been forgotten.

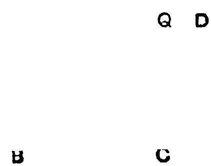
It will perhaps be wise to drive home the fundamental idea by finding the area of a circle drawn on squared paper. In any

case the teacher should constantly harp back on the fact that the fundamental way of finding an area is counting the number of times that it contains a standard unit of area. There is great danger that this fundamental idea will be forgotten, as the rules we shall use do not bring the fundamental idea into prominence.

Now ask the class what is the easiest area to find. Some will say a rectangle, some a square. The teacher may point out that the square is only a special case, so that we can agree that a rectangle is the simplest area to find.

Ask the class what area they would like to consider next. Possibly they will suggest the right-angled triangle; but I should persuade them to consider the **parallelogram**.

Take a rectangular sheet of paper PBCQ. Cut off PAB and move it to the position QDC.



It is easy to deduce from this that the area of the parallelogram is measured by the product of its base and height.

A model of a parallelogram consisting of two equal rods with their ends joined with elastic is useful; the one rod may be held on the blackboard while the other is moved along a parallel line drawn on the board. It is interesting to notice the changes in the lengths of the diagonals and in the angles between them.

The **triangle** is easily dealt with by regarding it as half of a parallelogram or half a rectangle. Here again the class should watch the changes in a triangle with a fixed base as the vertex is moved along a line parallel to the base, and note that the area does not change.

The class must now have some practice in finding areas of parallelograms and triangles. They will have difficulties with drawing the altitudes of an obtuse-angled triangle; time will be saved if they use their scrap paper and draw a few altitudes freehand before they go on to exact work. Some boys will make the startling discovery that the altitudes of the acute-angled

triangle are concurrent; give them encouragement for this and let them try whether this is true of a right-angled triangle and an obtuse-angled triangle; finally hold out to them the hope that some day they will be able to prove it.

The **area of a trapezium** is so important in much later work that it is worth while to find it and lay stress on the rule:

"The area of a trapezium is measured by the product of its height and the average of the two parallel sides."

Of course the class must discover this for themselves; there are several ways of doing this*.

As to practical work, all that remains is to point out that the area of any rectilinear figure can be found by dividing it into triangles.

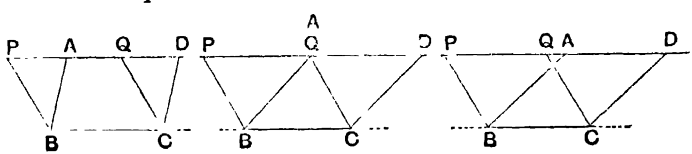
Now we must go on to the theoretical work.

We might naturally take first:

"A parallelogram and a rectangle on the same base and between the same parallels are equal in area."

But, while we are about it, we may as well take the following slightly more general theorem which is just as easy to prove:

"Parallelograms on the same base and between the same parallels are equal in area."



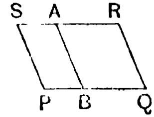
It is surprising how many boys, after proving

$$\triangle PBA \equiv \triangle QCD,$$

forget that, if they add the figure AQCB to each, their proof only applies to the first figure and it must be modified certainly for the third figure; whereas, if they take those two triangles in succession from the whole figure, the same proof applies to all possible figures.

* See S. and H., *J.G.* pp. 108, 109 or S. and H., *P.G.* p. 48.

The danger of misunderstanding, or incomplete comprehension of technical terms is well illustrated here. I have found boys argue that the two parallelograms in this figure are equal in area as they have a common base AB and are between the same parallels PQ, SR.



The class will now go on to:

"Triangles on the same base and between the same parallels are equivalent* (or equal in area)."

We may as well extend this to triangles on equal bases in the same straight line.

The converse may be mentioned, but the proof had better be left for a long time yet.

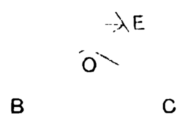
For the sake of shortening the proof of the theorem of Pythagoras we had better prove that:

"If a triangle and a parallelogram stand on the same base and between the same parallels, the area of the triangle is half that of the parallelogram."

A very valuable *vivâ voce* lesson can be given from this figure.

DE is given to be parallel to BC.

Ask the class to write down as many pairs of equivalent triangles as they can. Then discuss the matter *vivâ voce*.



Again, on another day, in the figure suppose that we are given that D is the mid-point of AB and E of AC.

Write down pairs of equivalent triangles.

Many points will arise out of the discussion.

These two lessons will drive home most of the necessary points in connection with equivalent triangles, and the class will be ready for writing out riders.

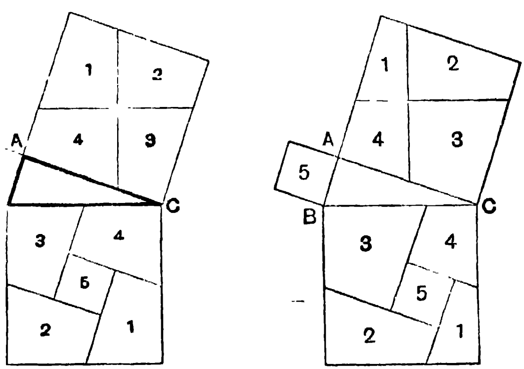
* The class should gradually get used to the word "equivalent" for "equal in area."

The construction for changing a quadrilateral into an equivalent triangle* must also be considered before going on to

THE THEOREM OF PYTHAGORAS

Probably some members of the class will know the result of this theorem already, but for some of the class a suitable introduction will be desirable. The result itself is so challenging that it will hardly be believed without proof; and the proof is hard enough to call for effort and not too hard, if divided up properly.

Tiled pavements and the 3, 4, 5 triangle are the simplest introductions to the theorem. If the class are then left to try it for any right-angled triangles which they may draw, interesting questions of degree of accuracy will arise. Perigal's dissection (the figure on the left below) will interest them; it is nice to have a cardboard model with the different parts coloured. The lines dividing the square on AC pass through the centre of the square and are parallel and perpendicular to BC.

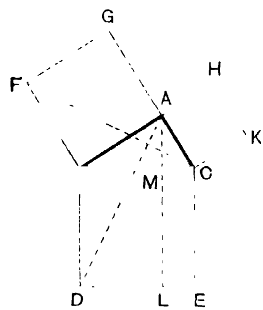


I have just discovered that the parallel and perpendicular to BC need not be drawn through the centre of the square; see

* See p. 296.

the figure on the right. Interesting developments from this arise from moving the point to different positions, but these distractions are not for the beginner.

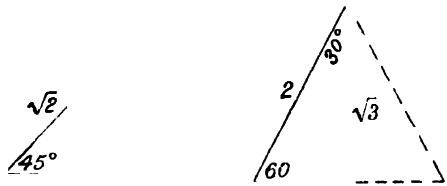
The ordinary Euclidean proof must be studied carefully if the class are to be able to reproduce it. At first it will save a lot of time, when they try to write it out, if they draw the figure first and have that passed before they go farther. To some extent the figure is a matter of memory, so I tell boys (i) always to draw it the same way up, (ii) to note that the largest square has to be divided into two parts, and I discuss which is more convenient for the proof, to draw **AML** parallel to **BD** or perpendicular to **BC**, (iii) I also point out that if the triangle **ABD** is turned through a right angle about **B** it fits on to the triangle **FBC**, and I shade these two triangles in my drawing on the board. Lastly I draw a figure with the angle **A** acute and consider how far the proof would apply to this figure and so emphasise the importance of proving **GAC** a straight line, a piece of the proof that boys often leave out.



The truth of the converse should be pointed out, but the proof may be postponed. The class is sure to be interested in the use of the converse for getting the right angles of a tennis court; they may well be told the story of the "rope-stretchers."

Pythagoras' theorem gives plenty of opportunity for numerical examples, in particular the class may become familiar with the two figures on the next page, which they will want later for trigonometry. This theorem also gives a golden opportunity for work in three dimensions, and I would strongly urge teachers to digress at this point and teach the class something about drawing a solid figure (see chap. XII).

The extensions of Pythagoras' theorem should certainly be postponed: they belong to a later stage.



THE AREA OF A CIRCLE

If there is time left over at the end of the term, it is interesting to consider the area of a circle*.

First the class may draw a circle on squared paper and count the squares Then they may draw a circle and a circumscribing polygon; and find the area of the polygon; this will lead on to the idea that the area of the polygon is $\frac{1}{2}r \times$ the perimeter of the polygon, and so to the area of the circle is $\frac{1}{2}r \times$ the circumference of the circle.

If time permits, there is the possibility with a good class of getting in some of the ideas which they may meet later in their mathematical careers under the name of "limits," though the word would not come in here.

* This will have the advantage of shortening the circle chapter which a long term's work.

CHAPTER X

THE CIRCLE*

CHORDS

If the class are not already familiar with the formulae for the circumference and area of a circle, they may as well learn them now (see p. 280). Then they will go on to consider the symmetry of the circle; they should be led to state for themselves the various theorems suggested by the figure of a chord and the perpendicular diameter. If they have been well taught and have caught the spirit of theoretical geometry, they will not only be willing, but actually keen, to prove the various theorems by congruent triangles; if they are bored by the proofs, they may well be left to the systematising stage (see chap. xv). They should also do some calculations of lengths of chords, etc., and the work should extend to three dimensions.

The construction for finding the centre of a given circle comes in here. It is essential that the given circle (or better, the given arc) on which the boy operates should not have been described by himself with compasses; otherwise he cannot be expected to see the advantage of a construction for a point already visible. He can draw his given arc by means of a guide curve such as the edge of his semicircular protractor or the lid of a circular tin. The fact that the centre is given by the intersection of the perpendicular bisectors of two chords will naturally be based on the chord-diameter property of the circle; but the opportunity should be seized of noticing that the matter may be viewed as an application of intersecting loci. To the mathematician the two ways of looking at the construc-

* If any of the class have not yet done the theorem of Pythagoras, it will be necessary to introduce them to that; it will be needed for calculations in connection with the circle.

tion are identical; but he must not forget that the boy's difficulty is rather to relate them than to distinguish the two methods.

Once again remind the class not to rub out the various arcs used in their construction. It is astonishing how frequently they are rubbed out.

TANGENTS

The formal definition of a tangent and the proof of the fundamental theorem present philosophical difficulties, and so do not belong to this stage of the work. By drawing a diameter of a circle and moving a line which is always at right angles to the diameter, the class can be led on to state the fundamental theorem; another method is to draw a chord AB, produce it both ways to P and Q and prove $\angle OAP = \angle OBQ$, where O is the centre, now move the chord farther and farther from the centre till A and B coincide These two methods should be enough for the class to realise the truth of the fundamental theorem. Numerical examples will follow. The equality of the tangents from a point will be obvious by symmetry, but the class will enjoy proving it as a rider.

ANGLE PROPERTIES

From the teacher's point of view this is one of the most fascinating little groups of theorems in the whole of geometry. The theorems themselves are surprising and the number of applications infinite; the group of theorems gives the boy a new sense of power.

There are various methods of attack. Here is a pleasant one.

Each boy draws a circle and takes two points A and B on the circle (not ends of a diameter). On the major arc let him take three or four points P_1, P_2, etc., and measure each of the angles APB; three or four points will do for the slower members of the

class, the quicker ones may take as many as they can in the time.

With a class that are all new to the work, I have never failed to find interest at once at the highest pitch. They will all find the various angles APB approximately equal in their own figure, and presumably they have all taken different cases—different sized circles and different positions of A and B—so that there is no need for further experiment. Still, they often like to see whether it is equally true for points Q (say) on the minor arc. (I am always guided as to whether to do this or not by the feeling of the class.) If they do it, some will also spot the sum of $\angle$APB and $\angle$AQB; I try to keep those quiet who spot it, so that each boy shall have the pleasure of discovering it for himself either now or later.

An enthusiastic teacher will now have the whole class eager for a proof.

Draw the first figure and ask for the connection between x and y. Many will get it without help, some will need a suggestion perhaps, or even a numerical instance.

Then take the second figure. Ask for suggestions and let each boy try it for himself.

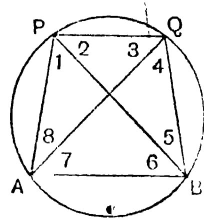
At this stage I should let them try to state the theorem and then write out the proof. After that, raise the difficulty presented by the third figure and discuss that case.

Now go on to angles in the same segment.

My experience is that boys spot the proof at once; many will see it for themselves before it is even stated.

"The angle in a semicircle is a right angle," and "The opposite angles of a cyclic quadrilateral are supplementary" will give the class equal joy.

And now for numerical applications. At first I always encourage boys to talk of "angles standing on the same arc AB"; later I get them to say "standing on the same arc, or in the same segment" and finally to use only "in the same segment." Some boys have difficulty in seeing which are angles in the same segment; I always take a figure like this (colour the four arcs with different colours and number the angles) and enquire "Which angle is held open by the arc AB?" ("the red arc," I should actually say), "Which arc subtends or holds open the angle QAB?" If a boy has difficulty, I make him hold a finger at each end of the arc and then name angles standing on that arc. Again, for the angle QAB, start at A and walk along each arm of the angle; what arc subtends the angle? The class soon tumble to the idea.



CYCLIC QUADRILATERAL

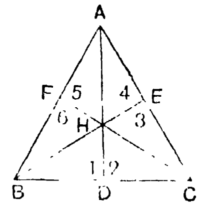
Then we take the theorem about the cyclic quadrilateral and prove it, and lay special stress on the corollary that "If a side of a cyclic quadrilateral is produced, the exterior angle so formed is equal to the interior opposite angle." I find so many boys that come to me do not use this as a piece of their machinery. It is also true that the theorem that "the exterior angle of a triangle is equal to the sum of the two interior opposite angles" is not as freely used as it should be.

The converses are difficult to prove: the proofs should not be taken now, but the class should be led to see these converses by intuition. Then there is a delightful lesson.

Every boy has his scrap paper ready and I draw on the board the figure on the opposite page.

"Here we have the altitudes of a triangle and we will assume that they are concurrent, i.e. pass through a common point."

"Write down as many sets of four concyclic points in the figure as you can."



Most boys will get three, a few may get four; it is very exceptional in my experience to get more than four.

I ask some boy for one set. He will probably give A, F, H, E.

Master. "Why are they concyclic?"

Boy. "Because 4 and 5 are right angles."

Master. "That is not accurate enough. Is it because they are equal or because their sum is two right angles?" etc.

I then give a chance to those who have not found the two corresponding sets. If they have difficulty, I point out that AH is the diameter of the circle AFHE; I have had to go so far as to draw in the circle, but I try to make them imagine the circle. Then I ask for a line corresponding to AH and finally we get the sets B, D, H, F and C, E, H, D.

Then some bright boy has probably put down B, F, E, C. "Why are they concyclic?" "What is the diameter of that circle?" Then we all try to find the corresponding circles on CA and AB.

Finally I mess up the whole figure by drawing in all six circles (freehand of course) and rub it out and the lesson is over

A week after I give the same lesson again; all will now be able to give me three sets at least, nearly all will give four, and the majority six. A little encouragement will bring all six; then we all write down our reasons for each of the six.

If the class is a bright one, I get them to prove that the altitudes of a triangle are concurrent—a gentle hint and a few leading questions will extract the proof from them, but I do not expect them to remember it.

Again, only if the class is very bright, I compare the figure for an acute-angled triangle with that for an obtuse-angled triangle, seeing which quadrilaterals are cyclic for the same reason as before and for which the reason has changed.

Finally, with a very bright class, I express all the angles in the figure in terms A, B, C by means of cyclic quadrilaterals; then join DE, EF, FD and do the same with that figure. This gives a very good lesson with a really bright class; I know it does not stick in the minds of many of them, but it opens up a vista; and, though it has not stuck, it has added to their power.

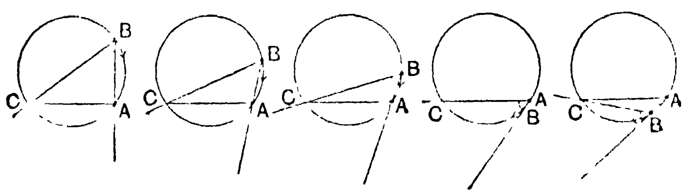
Another attractive lesson can be given from the figure for Simson's line; again the proof can be drawn out from a good class, but I should not expect it to be remembered.

ANGLE IN ALTERNATE SEGMENT

This gives another delightful lesson. First of all we consider the figures below (it will help to draw CB in blue and BA in red in each of the figures) and note that the angle B remains constant. By considering the first and last figures, we see that the theorem about the exterior angle of a cyclic quadrilateral is really the same theorem as that about angles in the same segment.

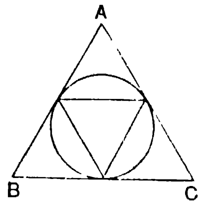
Then the fourth figure should, with a little help perhaps, suggest the alternate segment theorem.

The proof of that theorem is taken next; a little help will enable the class to find it for themselves.



It should be pointed out that in the proof we show that the angle between the tangent and the chord is equal to a *particular* angle in the alternate segment, but that all the angles in that segment are equal to one another, therefore the angle is equal to *any* angle in that segment.

Again, the result must be driven home by numerical examples, and finally the angles of this figure may be expressed in terms of A, B, C, and some interest can be aroused in the resulting proof that tangents from a point are equal.



CONSTANT ANGLE LOCUS

The construction of this locus must be taken, first when the angle is less than a right angle, then when the angle is greater than a right angle.

The class are pretty sure to get at the construction by considering the angle at the centre, which is a perfectly good way to do it. But they will probably appreciate the construction depending on the alternate segment theorem, which is really the best construction. I think the class should be taken on to this later method fairly soon, or else they will get the other method so ingrained that they will continue to use it instead of the better method.

CHAPTER XI

RIDERS

THEIR IMPORTANCE

In the first term of geometry, the teacher's main business is to give the boy certain fundamental conceptions and some geometrical vocabulary.

During the rest of the first two years, the main business is the acquisition of geometrical facts and *the power to apply them.* During the rest of his school life, and afterwards, so far as he is breaking new ground, he must still be acquiring facts and the power to apply them. A fact that a boy says that he knows but cannot apply can hardly be said to be a real possession.

It seems, then, that the real aim in teaching geometry should be to give power, and not to train his memory. The mere learning of the proofs of theorems will not take the boy very far; the real training must come from applying the facts he has. After all, the proving of theorems is merely the application of facts he has already acquired; but the learning of theorems is rather passive, it is like watching someone else apply known facts; whereas rider work is active, the boy is doing the application himself. There is no doubt that rider work will develop the boy's power more than learning model proofs.

I do not want to disparage the learning of proofs of theorems, about the age of 15 or 16 that becomes important; but certainly during the first few years of geometry the acquisition of power by means of riders is far more important, and that work will make the learning of proofs of theorems a much simpler matter when the systematising stage is reached.

METHODS OF ATTACK

The first thing is to translate the words of the book into a figure and to mark in that figure all the relevant data.

The boy should draw his own figure. The figure should be a fair size and reasonably neat; straight lines should look straight, right angles right, and parallel lines parallel; compasses may be used for circles, if needs be, but the use of a ruler should in general be forbidden. He should also mark in the figure what is given; and then the master should draw his figure on the board and see that all the data are marked. The things to be proved equal may be marked with a query.

It is a good plan to make the boy mark his data in ink, leaving pencil marks for his inferences or deductions. In the same way the master should mark his data in white chalk and his inferences in some other colour; there is something to be said for using various colours for the data and white for his inferences; but the white chalk seems to correspond to the boy's ink.

Of course the same marks should be used for things that are equal. Right angles are best marked as in the figure. The use of 90° should be avoided as much as possible in theoretical work. Parallels may be marked as in the figure.

Throughout a boy's work at geometry right up to the School Certificate stage he will be doing some riders which I call "one-step" riders, i.e. riders which can be proved by the application of a single theorem; even the weakest boys can be taught to do riders of this type.

In this stage of rider-work the boy should be taught when in difficulties to ask himself:

(i) What does the data tell me?

(ii) What theorem do I know that brings in the terms of the question?

(iii) Is there any piece of the data that I have not used?

At yet a later stage the boy should be taught to use "synthesis" and "analysis" and to combine the two.

SYNTHESIS, ANALYSIS

In the synthetic method the boy goes forward from his data to what he wants to prove. This is the ordinary method adopted in writing out in finished form the proof of a theorem or rider, but it is not the most powerful method, nor the most usual method, of *discovering* the proof of a long difficult rider.

In the analytical method the boy takes the result which he wishes to prove and says "This will be true if I can prove so and so," then he goes back and tries to prove that, and says "It will be true if I can prove something else." In fact we may say that in the analytical method the boy works backwards from what he wants to prove to what he is given.

Boys should be encouraged quite early to try the analytical method; it may have little success at first, but it makes a strong appeal even to the weaker boys, and may give them a useful start.

In actual practice, in easy riders a boy uses the synthetic method, but in harder riders he should be trained to use a combination of the two.

First of all, keeping an eye on what he wants to prove, he should mark on his figure in pencil inferences from his data. When he has gone as far as he can, he should look at what he wants to prove and say "To get this all I want is so and so." Gradually working from both ends his arguments will, as it were, meet in the middle. Then he will write out his proof in the usual synthetic form.

LEARNING TO USE NEW THEOREMS

When once a boy can do riders involving application of parallels and the congruence theorems he has acquired a useful power, and he likes using it. The next difficulty is to get him to use the new weapons put into his hands; he seems always inclined to try to use congruent triangles for everything. I tell him that as a carpenter he first learned to use a pocket knife; later he is taught to use a brace and bit; he would not think at that stage of using a pocket knife for every job he had, so in geometry he must try to use the new tools he gets.

After a class has mastered the set of theorems about a parallelogram, let the master set this simple rider "ABCD is a parallelogram; X, Y are the mid-points of AB, CD Prove DX parallel to YB." He will probably find that at least half the class try to do it by congruent triangles, instead of by a simple application of one of the converses of the parallelogram theorems.

It is often useful to pick out a set of riders, which can be done either by the use of congruent triangles or by later theorems, and to tell the boy that he must solve them without using congruent triangles *directly**.

I would advise masters teaching at this stage to choose riders suitable for this purpose and mark them in their books.

This tendency to use only a few theorems extends right through the course of elementary geometry, and it must be watched and corrected.

E.g., boys seldom use freely "The exterior angle of a triangle is equal to the sum of the interior opposite angles," or "The

* I remember a boy coming to me once with the rider "A straight line ABCD cuts two concentric circles in A, B, C, D, prove that AB = CD." I suggested "If you draw the perpendicular from the centre on to ABCD, what does that do?" The boy said "Oh yes, I can do it like that, but it depends on congruent triangles." I explained that he had missed the point and ignored the force of the word "directly."

exterior angle of a cyclic quadrilateral equals the interior opposite angle."

Again, in a triangle with a parallel to the base, they tend to rush into similar triangles when the fact that the sides are divided proportionally is simpler and shorter to use.

LESSONS ON RIDERS

I want to discuss the methods of attacking a few riders. As most new theorems should be attacked as though they were riders, I make no excuse for taking first a rider of the days of my youth that has been glorified into the rank of a theorem.

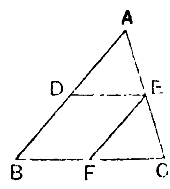
The straight line drawn through the middle point of one side of a triangle parallel to another side bisects the third side.

There are two possible methods of procedure:

(i) Fire the enunciation at their heads and let them draw their own figures; ask for suggestions for some more construction, and so get at the proof as given in most books.

(ii) Let the class invent the rider for themselves. Let me sketch a lesson which I have often found enjoyable.

I draw a triangle on the board and tell each member of the class to draw his own triangle and say "I am going to draw a line parallel to the base. Where shall I draw it?" This will raise various suggestions, but I choose the suggestion that pleases me. "Half way down" is a good suggestion, but we must be precise as to what we mean by that. Someone will probably suggest drawing it through the mid-point of AB (see the figure). We all draw the line in our own figures. Probably someone will say "But that goes through the mid-point of AC," and there I have what I wanted. "But do we know that it must go through the mid-point of AC? Do you all feel pretty sure that it does? Then let us try to prove it."



So far we have only arrived at the stage at which (i) above

starts; but the few minutes spent on getting the class to state for themselves what we are going to prove adds to the interest.

The first thing is to mark in the figure all that we know or can deduce from the data, viz.

$$AD = DB, \quad \angle ADE = \angle B, \quad \angle AED = \angle C.$$

Can we do anything with a figure like that? We must make some construction. I generally find that someone will suggest drawing (*a*) a parallel to AB through E, or (*b*) a parallel to AC through D.

Either of these constructions will give a nice easy proof and both should be discussed. Then perhaps we might discuss the proof obtained by drawing (*c*) the parallel to BA through C and producing DE to meet it at F; another quite good proof.

Then we might consider the converse.

The straight line joining the middle points of two sides of a triangle is parallel to the third side.

Naturally we shall try the constructions (*a*) and (*b*) above and we shall find difficulties with them, so let us try (*c*) and we shall find that we can prove it.

Such a lesson I always find attractive, and I believe the class enjoy it. The next out of school work should be to write out the proofs of the two theorems.

Here is another rider (S. and H., *T.G.* p. 50, No. 38). T, V *are the mid-points of the opposite sides* PQ, RS *of a parallelogram* PQRS. *Prove that* ST, QV *trisect* PR.

Let the class draw their own figures and give them a few moments to settle their plan of attack. Some of them are almost sure to try it by congruent triangles, so do not leave them too long at it. Let us bar the congruent triangles.

Suppose PR cuts ST, QV at X, Y.

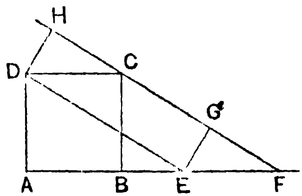
"Is there any piece of your figure that looks like the figure of any theorem that you have done?"

If that fails to draw, "Concentrate on the triangle RXS", perhaps even draw it in a different colour.

It will be a very dull class in which no one suggests that we should get XY = YR if we knew YV to be parallel to SX. "Well, let us try to prove that, or ST parallel to QV. What do we know in the figure STQV?"

S. and H., *T.G.* p. 58, Ex. 50; or S. and H., *J.G.* p. 129, Ex. 59. *In the figure* ABCD *is a square and* DEGH *is a rectangle. Prove that they are equal in area.*

Here again the class should draw their own figures and try it each for himself. Again some will try and rush into congruent triangles.



"What theorems do we know about areas?" or even "What is the first theorem we had about areas?" must produce "Parallelograms on the same base, etc."

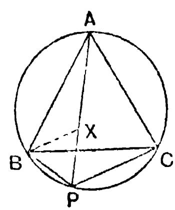
"What parallelograms are there in the figure?"

Colour them, if needs be, in the figure on the board.

"Can you see two parallelograms on the same base and between the same parallels?"

"Can you find another pair? Turn your figures round, if needs be," etc., etc.

S. and H., *T.G.* p. 103, Ex. 36. ABC *is an equilateral triangle inscribed in a circle;* P *is any point on the minor arc* BC. *Prove that* PA = PB + PC.



Most boys should be able to suggest "From PA cut off PX = PB." Then XA has to be proved equal to PC.

Congruent triangles are sure to be suggested. The triangles must be XA ..., PC .... There is not much choice for the third corners. Most boys will go for △XAB; △PCB is possibly a little more difficult, but any other suggestion can easily be shown to be hopeless.

Everyone should get $AB = CB$ and $\angle BAX = \angle BCP$.

Now $AX = PC$ is what we want to prove, so the third thing must be a pair of equal angles.

The class will easily get that $\angle ABX$ would equal $\angle CBP$ if only $\angle PBX$ equalled $\angle ABC$ which is $60°$.

Then we have to prove $\angle PBX = 60°$, so we must prove ΔPBX equilateral. We already know $PB = PX$, so we have only to get $\angle PBX = 60°$; is it equal to any angle of ΔABC? Why?

S. and H., *T.G.* p. 101, Ex. 48. *If* A, B, C, D, E, F *are six points in order on a circle. Prove that*

$$\angle A + \angle C + \angle E = \angle B + \angle D + \angle F = 4 \text{ right angles.}$$

It is rather nice to suggest considering what arcs subtend the various angles. But I am rather fond of drawing AD and telling them to mark the one set of angles. "What do you know about any of the angles you have marked?"

This will generally produce a good crop of solutions.

CONSTRUCTIONS

It should be pointed out that in many cases of construction the problem reduces itself to finding a certain point, and in nearly all such cases the point is found by finding the point of intersection of two loci on both of which the point must lie.

For example, in bisecting a straight line AB, the required point must lie on AB and also on the locus of points equidistant from A and B.

Again, in finding the circumcentre of a triangle ABC the centre must be equidistant from A and B, and also equidistant from A and C.

This aspect of many of the familiar constructions should be stressed.

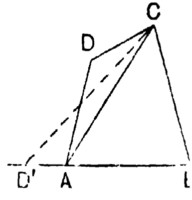
The class should be taught that, when a construction has to be made, it is often useful to draw the required figure by eye, and

then to study the properties of the figure and so get ideas for making the construction.

A very nice instance of this is the construction for *the common tangent to two circles*. I find so many boys are shown this, instead of being led to discover it; they then regard it as a piece of jugglery to be remembered, and often they make mistakes just because they trust to memory and it plays them false. I always draw two circles, centres A and B say, and draw by eye a common tangent ST. "Now imagine that the two circles are two rotating discs and that ST is a planing machine that cuts both discs away at the same rate." Draw out from the class that ST will move parallel to itself. "What will happen ultimately?" This leads pleasantly to constructing the circle with radius equal to the difference between the radii of the given circles; also it leads up to the idea of parallel translation which will be of use later. For the interior common tangent, we have to imagine a wonderful machine that planes at one end and adds on stuff at the other, but the class enjoy that.

This sort of thing should not merely be done with harder constructions such as I have considered above; even with easy constructions the class should be led to discover them for themselves; all the circle constructions (e.g. the circumcircle of a triangle) lend themselves particularly well to this treatment

With the construction for *a triangle equivalent to a given quadrilateral* I say "the figure represents a section of a cliff and the part ADC slides down to the beach below. If D goes to D′, what do we know in the figure?"



LOCI

In finding a locus the boy should be taught to start by plotting some points on it, and so finding out what the locus is going to be.

In more advanced work he should learn to look at special cases and special points on the locus. Here is quite an advanced case, but it illustrates what I mean.

A *is a fixed point outside a fixed circle,* BC *is a variable diameter of the circle; what is the locus of the orthocentre of the triangle* ABC*?*

First take the special case in which BC passes through A; from that the locus goes to infinity, so it is probably a straight line. Then take the special case in which AB touches the circle, from that the points of contact of the tangents from A are on the locus, so the locus is probably the polar of A. This preliminary investigation is a great help to finding a solution of the question.

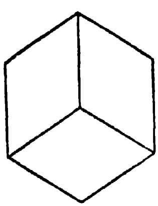
CHAPTER XII

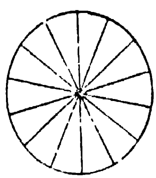
THREE-DIMENSIONAL WORK

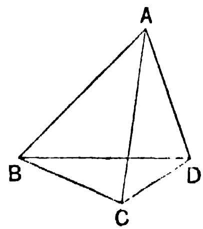
SEEING A SOLID FIGURE

Some boys have considerable difficulty in visualising a solid figure from a two-dimensional drawing of the figure. It is not an uncommon experience that boys of 16 or 17, who have had no training in solid geometry, find this much more difficult than do boys of 13 or 14. Whether the work I am going to suggest should be done at a still earlier age I cannot say; but I strongly advocate that by 14 a boy should have had some lessons in solid geometry, as suggested below.

The first thing is to get the boy to lift (mentally) certain lines in a plane figure out of the plane, so that he sees a solid figure. I have found the following figures very helpful, and give them in the order in which I have found boys take them most easily.







Now each of these figures can be made to represent a solid in two different ways.

The first figure represents either a cube or three planes with their common point away from the spectator (as though he were looking at the corner of a room). The boy has little difficulty in seeing the cube; he may be helped to see the other by standing a half-open book on the table and fitting a piece of cardboard in the enclosed space on the table.

As soon as all the class see both solids, I make them look at my (very rough) drawing on the board and tell them to bring the centre point towards them, then to move it away, then bring it back again. I make them do this several times. This I call "eye-gymnastics": it is training the mental eye to lift the plane figure into a solid figure.

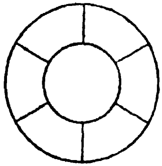
In the same way I use the second figure; it represents a bell tent looked at from above or from below the ground; again I make them change it backwards and forwards several times.

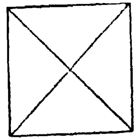
Again, the third figure represents a tetrahedron with AC in front or with BD in front. This boys often find distinctly hard, and I sometimes find it necessary, when they want to get BD in front, to make AC fainter and BD heavier, or to rub out part of AC where it crosses BD. Again they change it backwards and forwards several times.

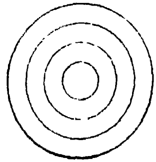
I am not at all sure that the master who is a skilful artist will do this work as well as a master whose drawing is rough and ready. The artist with his beautiful figures may find it easier to get the boy over his initial difficulties; but the boy must be able to visualise a solid from his own figures, so that he should have practice with blackboard figures that are quite simple and rough.

Models no doubt may also be helpful for complicated figures, but dependence on models for simple figures merely saves the boy from using his imagination, and so makes it more difficult for him to work with a plane figure when a model is not available.

Other figures that have been suggested to me for "eye-gymnastics" are:







A lampshade, a frustum of a cone: probably better than the cone already considered.

A pyramid on a square base.

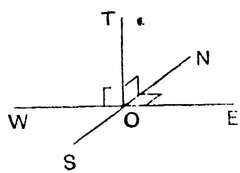
Either a cone or a sphere.

DRAWING A SOLID FIGURE*

The following points are a great help to the boy in drawing solid figures. His attention may be drawn to them in pictures or in drawings.

(i) All lines that are vertical in a solid figure should be represented by lines that are parallel to the side edges of the paper.

(ii) Horizontal lines are not necessarily parallel to the top and bottom edges of the paper. See the figure†, which represents an East and West road crossing a North and South road with a vertical flagstaff OT at the crossing.



(iii) In a drawing of a solid figure, in general, a rectangle looks like a parallelogram, a circle looks like an ellipse, right angles look like acute or obtuse angles.

On p. 230 we have already seen how to make a simple drawing of a box.

In drawing a figure of a pyramid or a tetrahedron, first draw the base, which should be well foreshortened, then draw the axis of the figure, and finally draw the sloping edges.

These rules are generally enough to enable a boy to draw simple figures that are good enough to help him in solving three-dimensional problems in geometry and trigonometry.

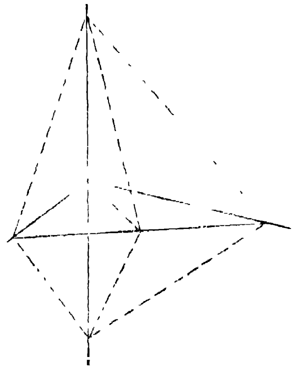
Some slight knowledge of plan and elevation is also helpful; but, as a rule, a boy should try to draw a general view of any solid which he has to consider, he may then draw separate

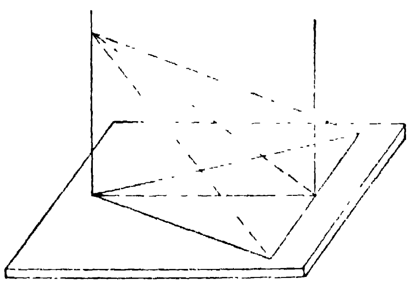
* See also G. and S., *Solid Geometry* pp. 89–92.

† Note particularly the method of indicating the right angles. I have only lately adopted it in three-dimensional work, but find it most helpful

figures of the various faces or planes of the figure, lettering them in the same way, of course, as in the general figure.

Another thing I have found helpful is to build up with spiked rods* a solid figure like this; then to let each boy make (i) a drawing of it as he sees it, (ii) a drawing from any position that he chooses. At a subsequent lesson I build up the same figure, dismantle it and *then* let them make their drawings. At a third lesson I describe the figure (without a model) and then let them make drawings again.





Yet another lesson which I have found helpful is to describe in words the above figure; I draw the figure in the air and then tell the class to draw it on paper.

This work is very attractive and the interest should be spread out by taking it in small doses; in that case, too, boys probably practise it in their own time.

* Made by Mr Geo. Cussons, Manchester.

CHAPTER XIII

GEOMETRY AFTER THE SECOND YEAR

There is still the ground of similar figures to be broken; besides that, a candidate for the School Certificate will have to do the "Systematising Stage," which consists of the whole ground taken so far, filling in many gaps and building it up into a logical whole

A preparatory school should leave the systematising stage alone. It is not so suitable for the preparatory school age; and, further, it is best to leave it to the public school so that there shall be no question of the boy being confused by learning two different orders for the theorems. The preparatory school master must decide whether to take similiar figures next; or to turn the boy on to revision of the ground already broken. My advice would be to revise the work already done, except in the case of exceptional boys. It is much more important that a boy going to a public school should be thoroughly sound on the work he has done than that he should have covered a lot of ground.

In a secondary school, the problem is rather different; the man in charge must decide whether revision is necessary or whether the class is ready to go on to similar figures and the systematising stage.

I have already referred to the difficulty caused by removes being made every term. This may be met in the next year's work by several consecutive divisions taking similar figures, say, in the autumn term, and a definite part of the systematising stage in the spring term, and another part in the summer term At the end of a year in those divisions, a boy will have covered the whole ground, whichever term he first joined one of the divisions.

CHAPTER XIV

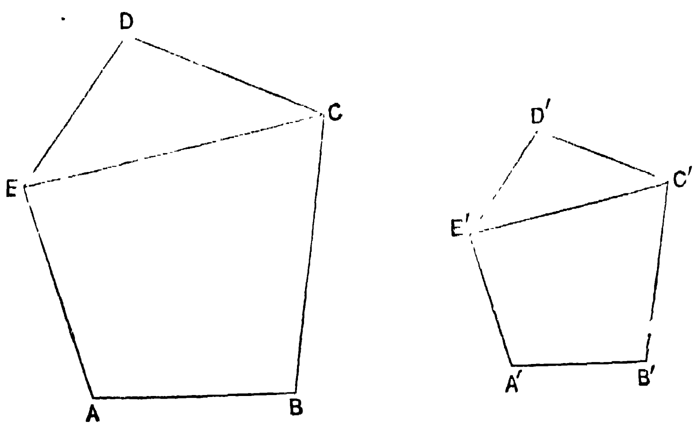
SIMILAR FIGURES

There is no reason why this group of theorems should be taken after areas and the circle, though that is the usual plan. A child has an idea of similar figures and has used it whenever studying a map or a plan, and of course in scale drawing.

The teacher's task is to bring into the conscious plane and to develop ideas which the child already has.

First of all it will be well to revise "ratio." The consideration of incommensurables is not required in any School Certificate examination. Such revision of a subject which a child has already dealt with in arithmetic is very valuable; the child will appreciate the work and see its significance much better in this revision.

The next question is: What are the essential properties of two similar figures? Naturally the teacher will refer to two maps, but he will simplify the question down to, say, two similar rectilinear figures. I prefer to take the two figures below.



With a little judicious questioning, the class will see that for two figures to be similar:

(i) the ratios of corresponding sides must be equal, and also

(ii) corresponding angles must be equal.

That (ii) by itself is not enough is made evident by comparing a square and a rectangle. That (i) by itself is not enough can be seen by imagining the lower part of one of the figures on the last page to be made of jointed rods and moved to the left or right.

Point out that these two properties are true of maps, plans, and scale drawing, as well as of the figures considered above.

SIMILAR TRIANGLES

In the case of congruent triangles, we have seen that we need not be told that all the corresponding parts are equal; it is sufficient if we know that three (suitably chosen) ones are equal.

The question naturally arises, How much is it sufficient for us to know about two triangles to be sure that they are similar?

Some discussion with the class should readily lead to the three simple sets of conditions. I do not propose to enlarge on the necessary discussion, though it is not always well done in schools; the teacher should study the text-book carefully and follow that.

As soon as these three theorems have been made clear by intuition (I do not advocate attempting logical proofs at this stage), a certain amount of numerical application is desirable, and then there are a lot of easy riders to be done. The class generally enjoy these and are very successful with them.

A TRIANGLE AND A LINE PARALLEL TO ITS BASE

There are still the theorem that a line parallel to the base of a triangle divides the sides proportionally, and its converse; also the problem of dividing a straight line in a given ratio.

I should be inclined to base these for the present on intuition; the proof of the direct theorem even for commensurables is not easy, and it belongs to the systematising stage. The results are very useful for purposes of calculation and they help to drive home the idea of ratio*.

THE RECTANGLE PROPERTIES OF THE CIRCLE

These follow from easy applications of similar triangles and give results which are surprising.

AREAS AND VOLUMES OF SIMILAR FIGURES

These need plenty of stress, and, besides providing a lot of numerical examples, lead to important ideas of variation References to engines and models should be brought in.

* See chap. XI, p. 292.

CHAPTER XV

THE SYSTEMATISING STAGE

At about 15 the average boy is ready for the systematising stage; the clever boy earlier; the boy with no turn for abstract thought perhaps never.

The boy should be told that he is now meeting a somewhat new type of thought. He will admit that, though he knows a fair number of facts and can prove a good many theorems, he has not arranged them very neatly in his mind and is probably rather uncertain as to what has been proved and what assumed. All this has to be cleared up: he has to systematise his knowledge. Intellectual curiosity, the prime motive hitherto, will not find much food in this stage; it must be satisfied elsewhere; perhaps new ground is being broken in trigonometry, mechanics or calculus.

I shall not say much about this stage. Choose a good book and follow it.

The chief difficulties arise in the theorems about the angles made by parallel lines and a transversal, and about congruent triangles. The difficulties are philosophical, and belong to the university stage, not to the school stage. Happily, most bodies that examine for School Certificates recognise this, and state in their regulations that proofs of these theorems will not be required; they may be assumed as facts to be built on. But the boy must be clear what these theorems are, and that they form part of the fundamental assumptions on which he is going to build; he must realise that, if any one of these assumptions is untrue, then his whole logical structure will fall to the ground It may interest him to know that there is a whole system of geometry that can be built up in which Theorem 5 and Playfair's axiom (on which it depends) are assumed to be untrue, and that

consequently many of the theorems which he will prove are untrue in that system of geometry.

There will be gaps to fill up, notably

THE INEQUALITY THEOREMS

These theorems may have been discovered in the earlier stage, but were of hardly sufficient interest then to demand proof. Now the proofs must be learnt, and the theorems about the greater side and greater angle of a triangle may be considered in connection with the sine rule.

"Any two sides of a triangle are together greater than the third side" does not appear in most examination schedules, but perhaps deserves a place if only for Heath's note*, which shows that the Greek mathematicians were not above a joke.

CONVERSE THEOREMS

The relation between a theorem and its converse has not been stressed in the earlier stage. The main point is that the converse of a true theorem is not necessarily true.

In many cases it is easy to establish a converse by an independent proof, and the memory need not be charged with the order of the two theorems. In other cases it is easier to proceed by *reductio ad absurdum* (the Greek phrase meant "reduction to impossibility"); there are variations of this method, namely "the method of exhaustion" (Theorem 17) and "the method of coincidence" (Theorem 27).

THE EXTENSIONS OF PYTHAGORAS' THEOREM

In the enunciation of these two theorems, a class sometimes has difficulty in remembering which has "plus" and which has "minus." Let the teacher hold two rods at right angles to

* T. L. Heath, *Euclid's Elements*, vol. I, p. 287. Quoted in S. and H., *T.G.* p. 38.

represent the two sides of a triangle and remind the class of Pythagoras' theorem; then consider what happens to the line joining the ends of the rods as the angle is made obtuse or acute. This should settle for ever the question of the "plus" or "minus."

If the class have already done some trigonometry, they will enjoy seeing that the two extensions can be united into the single formula $a^2 = b^2 + c^2 - 2bc \cos \mathsf{A}$; this will help them to write down the enunciations of the two extensions and to apply them to a particular figure.

If they have not done trigonometry, it is necessary to harp on the projection of a line, and to lay stress on the fact that the required rectangle is contained by one of the two sides about the angle and the projection on it of the other of these two sides

A useful exercise can be made from the figure of an acute-angled triangle ABC and its three altitudes:

$$\mathsf{AB}^2 = \mathsf{BC}^2 + \mathsf{CA}^2 \ldots,$$
$$\mathsf{AC}^2 = \mathsf{BC}^2 + \mathsf{AB}^2 \ldots,$$
$$\mathsf{BC}^2 = \mathsf{AB}^2 + \mathsf{CA}^2 \ldots;$$

fill up each gap in two different ways. Repeat with an obtuse-angled triangle.

As soon as the class are efficient at using the extensions they may go on the Apollonius' theorem, the proof of which they should easily evolve with a little help.

LOCI

The class must be made to realise that in proving a locus theorem there are really two theorems to be proved. Suppose that we wish to prove that the locus of a point moving under certain conditions is a certain line which we will call l, then it is necessary to prove (i) that every point satisfying the given conditions lies on l, (ii) that every point on l satisfies the conditions.

"In exercises on loci, in an examination, the double proof is not usually required, and unless it is explicitly demanded the

candidate is in a difficulty, and the examiner finds it impossible to make an adequate allowance for an answer that is really complete. As a matter of tactics, the examinee, faced with a rider in which the form of the question gives no guidance, is advised to limit his formal proof to one aspect of the locus and to mention that a converse proof is necessary to make the answer satisfactory. But it is highly desirable that examiners in setting locus problems should frame their questions to elicit exactly what they require, stating definitely which of the converse proofs they expect, unless they do want the double proof.

There is no reason why the teacher should allow the double aspect of the locus to be ignored. The logical error is serious, and in the vast majority of cases one half of the complete proof can be inferred from the other, either by *reductio ad absurdum* or by the perception that each individual step expresses not a one-sided implication but a reversible equivalence*."

THE USE OF LIMITS IN PROVING THEOREMS

When we are in the stage of discovering new theorems (e.g. about tangents), it is perfectly legitimate to call to our aid the idea of limits; again it is perfectly legitimate to use the idea for linking together various theorems and showing that they are merely different aspects of one general theorem. But the question will be asked, Can we use limits for the proofs of tangent theorems? To this I would say that the philosophical difficulties involved in the idea of a limit are such that it is better to avoid the use of limits in formal proofs and to leave them for the mathematical specialist†.

* See "The Teaching of Geometry in Schools," *Mathematical Association Report*, 1923, pp. 63–65.

† See "The Teaching of Geometry in Schools," *Mathematical Association Report*, 1923, pp. 44–48.

METHOD FOR THIS STAGE

To my mind, it is of little use to make boys learn theorem by theorem. They should already have logical power and, besides knowing the proofs of many theorems, should know the general form of a proof; as soon as they see the general outline of a proof, they should be able to write out the detailed proof for themselves. The main work should be looking at a whole group of theorems and seeing how they hang together.

There are two ways of revising a group of theorems (e.g., the angle properties of a circle):

Either What is the fundamental theorem of the group?
How do we prove that?
What theorem hangs on that?
How do we prove it? and so on.

Or What is the last theorem of the group?
How do we prove that?
What theorem or theorems does it depend on?
Then treat those theorems in the same way.

The essential of the systematising stage, to my mind, is to look at the structure as a whole, or look at large parts of it at once. The boy's earlier training should have taught him how to lay the individual bricks; now he wants to agree about the concrete foundation on which he is going to build, the definitions and assumptions, and then look at the whole structure.

A FIXED SEQUENCE, IS IT DESIRABLE?

Ever since Euclid was dethroned, some teachers and examiners have been crying out for a fixed sequence: having thrown off one set of fetters, they want to put on another set. The motives of this reactionary tendency have been clearly stated, and met, by Prof. Hobson.

"There are at the present time some signs of reaction against the recent movement of reform in the teaching of geometry It is found that the lack of a regular order in the sequence of propositions increases the difficulty of the examiner in ap-

praising the performance of the candidates, and in standardising the results of examinations. That this is true may well be believed, and it was indeed foreseen by many of those who took part in bringing about the dethronement of Euclid as a text-book. From the point of view of the examiner it is without doubt an enormous simplification if all the students have learned the subject in the same order, and have studied the same text-book, but, admitting this fact, ought decisive weight to be allowed to it? I am decidedly of opinion that it ought not. I think the convenience of the examiner, and even precision in the results of examinations, ought unhesitatingly to be sacrificed when they are in conflict—as I believe they are in this case—with the vastly more important interests of education. Of the many evils which our examination system has inflicted upon us, the central one has consisted in forcing our school and university teaching into moulds determined not by the true interests of education, but by the mechanical exigencies of the examination syllabus. The examiner has thus exercised a potent influence in discouraging initiative and individuality of method on the part of the teacher; he has robbed the teacher of that freedom which is essential for any high degree of efficiency*."

My own feeling is that we are very unlikely ever to have a fixed sequence; some agitators cry out for it, but examining bodies seem very well content with the formula "any proof of a proposition will be accepted that appears to the examiner to form part of a systematic treatment," and there is reason to believe that examiners interpret the rule in a liberal spirit. There is no chance of agreement as to the ideal sequence. The main point is that the boy should eventually have *some* sequence in his head. Even if teachers could agree on a sequence, it would be a mistake if such sequence were imposed on schools, for this would destroy the possibility of experiments that might lead to improvement.

* Presidential Address to Section A of the British Association, *Nature*, September 1910.

PART VI

FURTHER MATHEMATICS

BY A. W. SIDDONS

WITH A NOTE BY THE LATE
PROFESSOR CHARLES GODFREY

FURTHER MATHEMATICS

A quarter of a century ago comparatively few boys, except the mathematical specialists, did any mathematics beyond arithmetic, algebra and geometry.

Since those days much of the old arithmetic has been cut out, partly on the ground that a lot of it was unnecessarily technical, and partly because arithmetic now comes into almost every branch of school mathematics, and its applications there are more fruitful than the old work. Again, in algebra much heavy manipulation has been cut out, as quite unfruitful for any but mathematical specialists. Of geometry I will say something when I come to trigonometry.

The time saved by cutting down the arithmetic and algebra makes it possible to introduce practically all boys to some elementary work in trigonometry and calculus.

CALCULUS*

I do not propose to discuss the teaching of calculus, as I regard it as just outside the range of this book; but I would plead for its introduction before the specialist stage. If the ordinary work is limited to the differentiation and integration of simple powers of the variable†, boys of average ability can be given a good grasp of the principles of the subject and can use it for a large number of applications. Besides the value of the things which they do, they will get a vista opened up.

TRIGONOMETRY

Perhaps the biggest change in school mathematics is due to the dethronement of Euclid which took place about 1903. His books I–IV and VI contained some 150 propositions; today

* See pp. 41, 52, 53.

† See G. and S., *Algebra*, chaps. XXIII–XXVI.

these have been replaced by about 60 formal theorems. Of the time saved a good deal is taken up with the introduction of the new ideas as they occur, and their application to mensuration.

In connection with the latter it is natural to start trigonometry.

The trigonometrical ratios arise naturally in connection with similar triangles. It is a mistake to start by giving a class the six ratios at once. I would define the cosine first and then the sine, I should not introduce any other ratio for some days.

It is easy to show the class that the cosine and sine of any acute angle are each less than 1, so they will easily remember

$$\text{cosine} = \frac{[illegible]}{\text{hypotenuse}}, \quad \text{sine} = \frac{[illegible]}{\text{hypotenuse}}.$$

If they are then told that the two sides for the **cosine** **co**ntain the angle, they can easily write down sines and cosines. I find this method much better than any of the other tricks for remembering which is which ratio, and it helps in resolution in statics later. They must be tested by writing down the sines and cosines from right-angled triangles in various positions. Some practice in the use of tables must be given, and then they can do little problems involving sine and cosine, which will give practice in multiplication and division of decimals.

The early work in trigonometry may well replace some arithmetic, for it is only the application of arithmetic to new material.

They will be interested to hear that cos **A** was originally an abbreviation for sin (of the complement of **A**).

As soon as they are thoroughly familiar with the sine and cosine, but not before that, the class may be introduced to the

$$\text{tangent} = \frac{\text{sine line}}{\text{cosine line}}, \quad \text{cotangent} = \frac{\text{cosine line}}{\text{sine line}}.$$

This opens up a new set of examples.

With most classes the work up to this point should be entirely numerical, a great variety of examples can be taken from

geometry, bearings, heights and distances, buildings and mechanisms; a bright class, or the brighter members of a class, might do some symbolical examples.

What I have sketched above is perhaps enough for a first term with young boys.

In the following term the secant and cosecant* may be introduced and the use of the log sine, etc., tables.

In this term examples in three dimensions should be introduced (see "Geometry," chap. XII); and the class should learn that any triangle can be solved by drawing an altitude, but it is not desirable to grind them at solving triangles by that method, as it is better that they should learn to solve triangles by aid of the sine and cosine rules.

The sine rule should be taught and its easy application to logarithmic work should be stressed. Not only should it be thought of in its symbolic form,

$$\frac{a}{\sin \mathrm{A}} = \frac{b}{\sin \mathrm{B}} = \frac{c}{\sin \mathrm{C}},$$

but the boy should also think of it in words

$$\frac{\text{this side}}{\text{the sine of the opposite angle}} = \frac{\text{that side}}{\text{the sine of the opposite angle}}.$$

The cosine rule is valuable because it brings the two cases of the extensions of Pythagoras' theorem into a single formula, as well as because of its use in solving triangles.

In the course of the work with the sine and cosine rules it will be necessary to extend the meaning of the trigonometrical ratios so that they apply to at least obtuse angles; probably the best course is to extend them for angles of any magnitude. Here we have another instance of the way in which a definition over a limited range of values is extended to cover other values —an important idea in mathematics.

* As the reciprocals of cosine and sine. N.B. each ratio and its reciprocal have one "co" between them.

So much trigonometry should form part of any liberal education.

SECTIONS OF A CONE

[I found the following suggestive notes among the lat Prof. Godfrey's papers.

He described them as fragmentary notes and said that they might be useful to experienced teachers, though in the hands of beginners they might lead to desultory work. He added "Hard work is the first condition of successful mathematical teaching, but it cannot be doubted that the subject is apt to be heavy and to need a little yeast."]

As boys are familiar with cones, and meet with at least two of the conic sections in their graph work (parabola and hyperbola), it is quite legitimate, even with a low class, to bring these ideas into relation. Such raids into the territory of "higher mathematics" amuse the boys and often stimulate the imagination of a backward boy in an unexpected way.

Ellipse. They have probably drawn an ellipse with two pins and a bit of string. They should examine the limiting cases (i) taut string—linear ellipse, (ii) very long string—almost circular ellipse. Orbits of planets—two foci inside sun. What does focus mean? Why S and H? Greeks studied ellipse because apparently useless; irony of history when astronomy and nautical almanac were found to depend on this study.

(Limiting and special cases are generally worth study and constantly crop up in geometry riders; they illustrate the "kinetic" as contrasted with the "static" aspect of mathematics.)

Ellipse as **sun-shadow** of circle (show it). Talk about parallel projection generally; one system of parallels unchanged in length; all the rest changed with equal projection; lines of steepest slope reduced in ratio $\cos\theta:1$. Same effect might be produced on figure drawn on elastic paper which is stretched or

contracts vertically and is not deformed horizontally. Elastic squared paper might change parabola $y = x^2$ into parabola $y = 2x^2$; thus parabola may project into parabola. Parallelogram projects into parallelogram. When does rectangle project into rectangle?

Dissected cylinder (of which example should be shown) gives ellipse, another case of parallel projection.

Projection from a centre. Electric light shadow. What becomes of parallelogram? of a circle? Cone of shadow; shadow of billiard-shade on wall; our old friend the hyperbola. Dissected cone; ellipse, parabola, hyperbola, circle, two lines, one line. Double cone. Water line on conical buoy. Search lights give elliptical patch on horizontal cloud surface. Electric torch throws patch of light on wall; watch it change from circle to ellipse, to parabola, etc. Conical projection on parallel plane. Magic lantern; similar figures; linear dimensions; areas.

Parabola already known as a graph, probably forgotten, do it again. Path of cricket ball, fire-hose jet, orbit of comet; where is the other focus? Big Bertha; parabolic orbit above the zone of effective atmospheric resistance. Area of symmetrical parabolic segment $= \frac{2}{3}$ circumscribing rectangle (by counting squares). How to obtain a parabola as envelope of straight lines. Equation $y = x^2$ derived from focus-directrix definition. Illustrate with equation of circle from centre definition.

Hyperbola from equation again. $pv =$ const. Asymptotes (but be careful not to say that so-and-so happens at infinity).

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